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ON WAVES OF FINITE AMPLITUDE IN DUCTS

PART I. WAVE FRONTS

By R. E. MEYER

(*Department of Mathematics, The University, Manchester*)

[Received 2 May 1951]

SUMMARY

The shock-free flow of an inviscid, perfect gas in a duct of varying cross-section is investigated. It is assumed that the area of cross-section changes slowly enough for the variation of the velocity over any cross-section to be negligible.

When a steady, shock-free flow is set up in such a duct, and the pressure at the entry, or the exit, of the duct is then lowered, or raised (in such a way that a shock wave is not formed immediately), then 'wave fronts' are formed which separate the region of steady flow from the region of unsteady flow. It is assumed that the rise, or fall, of the pressure is initiated with a discontinuity in the local rate of change of pressure.

The equations of motion are integrated exactly on such wave fronts (sections 3, 4), and the tendencies for different wave fronts to form shocks, or decay, or to approach an asymptotic form are discussed. The results are summarized in section 5; they explain the failure to establish, in practice, steady shock-free flows decelerating through the sonic speed.

1. Introduction

In the theory of shock-free flow of an inviscid gas a new solution of the equations is obtained from any known solution if the sense of the velocity is reversed at every point. In particular, if a converging-diverging duct is such that a steady shock-free flow is possible in which the gas is accelerated from a subsonic to a supersonic velocity, then theoretically a shock-free flow in the opposite direction is also possible in which the gas is decelerated from supersonic to subsonic speed. The former (accelerated) flow will be called 'nozzle flow', the latter (decelerated) flow 'diffuser flow'.

In practice only nozzle flow has been observed, and from these observations it has been inferred that the corresponding shock-free diffuser flow is unstable. For the understanding of the mechanism of compressible fluid motion it is a matter of interest to determine whether this instability can be explained from the theory of inviscid fluid motion or whether boundary-layer effects must be taken into account in any explanation. Moreover, if the theory of inviscid fluid motion suffices, it is still of some interest to determine whether it is necessary to consider the full, two- or three-

dimensional flow† in such a duct, or whether it is possible to explain the instability by the help of the simple theory of plane waves in a duct of slowly changing cross-section.

The physical argument commonly advanced in favour of the last-named alternative is as follows. In a compressible fluid small disturbances are propagated with the local speed of sound relatively to the moving fluid, and any disturbance therefore gives rise to an 'advancing' wave and a 'receding' wave, which reach particles upstream and downstream, respectively, of the particles originally affected by the disturbance. As seen by an observer at rest, advancing waves always move downstream, whereas receding waves move downstream in supersonic flow and upstream in subsonic flow. In both nozzle flow and diffuser-flow, advancing waves originating downstream of the throat pass immediately out of the duct and those originating upstream travel through the throat and exit at high speed. Moreover, in nozzle flow, all receding waves move away from the throat, wherever they originate. On the other hand, in diffuser flow, all receding waves travel towards the sonic line, at which they accumulate asymptotically, since their absolute speed of propagation tends to zero as the local Mach number approaches unity. Thus only in diffuser flow, and not in nozzle flow, do the circumstances favour the accumulation of disturbances.

Attempts to extend this argument to a *proof* of the instability of diffuser flows were made by Brown (**1**) and Kantrowitz (**2**). However, their results appear to be contradictory, and the arguments themselves are open to serious criticisms (see Appendix).

In the following, attention is restricted to the fronts of the waves, where the disturbed region borders on the still unaffected, steady flow. It is assumed that the acceleration of the fluid particles changes discontinuously when the wave front passes over them; the case where the acceleration is continuous is treated in Part II, and leads to similar results.

In addition to nozzle flows and diffuser flows, the analysis applies to all steady shock-free flows where the speed is subsonic throughout the duct, or supersonic throughout the duct, including the extreme cases where sonic speed is just reached in the throat. It applies also to ducts without throat and to ducts with several throats.

2. The characteristic equations

Consider the unsteady, isentropic flow of an inviscid, perfect gas in a duct whose cross-section changes slowly enough for the velocity to be taken

† A stability investigation for two-dimensional, transonic flow has been carried out by Kuo (**5**).

as constant over any plane normal to the axis of the duct. The motion may then be described in terms of two independent variables, x and t, denoting respectively the distance along the axis and the time. The momentum equation may be written ((3), p. 28)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \tag{1}$$

where u is the velocity in the direction of x increasing,† ρ the density, and p the pressure. The equation of continuity is

$$Q\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u Q)=0, \tag{2}$$

where Q is the area of the cross-section. Since the flow is isentropic,

$$dp = a^2\,d\rho, \tag{3}$$

where a is the local speed of sound. When the dependent variables are expressed in terms of Riemann's variables,

$$\left.\begin{aligned} r &= \tfrac{1}{2}u+\tfrac{1}{2}\int\limits_0^\rho (a/\rho)\,d\rho = \frac{a}{\gamma-1}+\frac{u}{2} \\ \text{and}\qquad s &= -\tfrac{1}{2}u+\tfrac{1}{2}\int\limits_0^\rho (a/\rho)\,d\rho = \frac{a}{\gamma-1}-\frac{u}{2} \end{aligned}\right\}, \tag{4}$$

where γ is the ratio of the specific heats, the equations of motion, (1) and (2), are transformed into

$$\left.\begin{aligned} \partial r/\partial t+(u+a)\,\partial r/\partial x &= -\tfrac{1}{2}auH \\ \text{and}\qquad \partial s/\partial t+(u-a)\,\partial s/\partial x &= -\tfrac{1}{2}auH \end{aligned}\right\}, \tag{5}$$

where
$$H = (1/Q)\,dQ/dx,$$

and by (4),

$$\left.\begin{aligned} u &= r-s, & u+a &= \frac{\gamma+1}{2}r-\frac{3-\gamma}{2}s \\ a &= \frac{\gamma-1}{2}(r+s), & u-a &= \frac{3-\gamma}{2}r-\frac{\gamma+1}{2}s \end{aligned}\right\}. \tag{6}$$

In the plane of the variables x and t (Fig. 1) the lines given by

$$dx/dt = u+a$$

and

$$dx/dt = u-a$$

are called *advancing* and *receding Mach lines*, respectively. In this plane, ordinary differentiation with respect to time along a line of slope

$$dt/dx = \theta(x,t)$$

† x is measured downstream from the throat (Fig. 1), and u is assumed to be positive throughout.

is equivalent to the application of the operator $\partial/\partial t+(1/\theta)\partial/\partial x$. The operators

$$\frac{D_+}{Dt}=\frac{\partial}{\partial t}+(u+a)\frac{\partial}{\partial x} \tag{7}$$

and

$$\frac{D_-}{Dt}=\frac{\partial}{\partial t}+(u-a)\frac{\partial}{\partial x}, \tag{8}$$

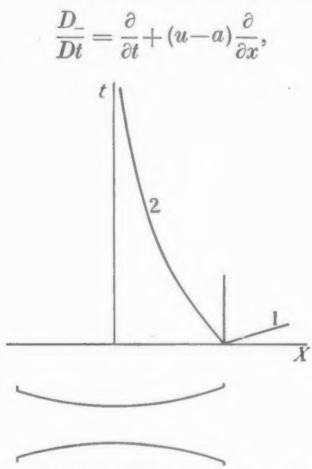

FIG. 1. 1. Advancing wave front. 2. Receding wave front in diffuser flow.

therefore represent ordinary differentiation with respect to time on the advancing and receding Mach lines, respectively, and by (5), the equations of motion may be written

$$D_+r/Dt = -\tfrac{1}{2}auH, \tag{9}$$

$$D_-s/Dt = -\tfrac{1}{2}auH. \tag{10}$$

The derivatives D_+s/Dt and D_-r/Dt, on the other hand, are not determined in terms of a, u, x, and t by the equations of motion without reference to particular boundary conditions, and these derivatives can be discontinuous. A differential equation for the variation of D_+s/Dt on a receding Mach line is, however, obtained as follows. By differentiating (10) with respect to time along an advancing Mach line, taking account of (6) and (9), one finds that

$$\frac{D_+}{Dt}\frac{D_-s}{Dt} = \tfrac{1}{2}(\gamma-1)H(\tfrac{1}{2}auHr+sD_+s/Dt)-\tfrac{1}{2}au(u+a)\,dH/dx. \tag{11}$$

Moreover, by (7), (8), (6), (9), and (10),

$$\left[\frac{D_-}{Dt}\frac{D_+}{Dt}-\frac{D_+}{Dt}\frac{D_-}{Dt}\right]s = \frac{\partial s}{\partial x}\left(\frac{D_-}{Dt}(u+a)-\frac{D_+}{Dt}(u-a)\right)$$

$$= \frac{\gamma+1}{4a}\left(\frac{D_+s}{Dt}+\frac{au}{2}H\right)\left(\frac{D_-r}{Dt}+\frac{D_+s}{Dt}+\frac{3-\gamma}{\gamma+1}auH\right). \tag{12}$$

By adding these two equations, a first-order, ordinary, *non-linear* differential equation is obtained for $D_+ s/Dt$ as a function of the time on a receding Mach line. The coefficients depend on the distribution of r, s, x, and t on this Mach line. A similar equation governs the variation of $D_- r/Dt$ on an advancing Mach line.

In the following, the unsteady flow through converging-diverging ducts is studied for which there exists a steady-flow solution without shock. This steady solution is given by Reynolds's 'hydraulic' theory and the velocity, local speed of sound, and Riemann variables in this solution, denoted by U, A, R, and S respectively, may be considered as known functions of x. They satisfy the equations (4) and (6) and ((**3**) p. 379)

$$\left.\begin{aligned} AUH &= (U-A)(U+A)U'/A \\ \text{and} \qquad A' &= -\tfrac{1}{2}(\gamma-1)UU'/A \end{aligned}\right\}, \tag{13}$$

where a dash denotes ordinary differentiation with respect to x of a function independent of time. It will be assumed that $H = (1/Q)\,dQ/dx$ is continuous and vanishes only at a throat, and that H' is bounded.

3. Receding wave fronts

Suppose a steady shock-free diffuser-flow has been set up in a converging-diverging duct, and assume that a disturbance is generated at the exit only, in such a way that the start of the disturbance is marked by a discontinuity of the acceleration (but not of the velocity itself). The disturbance spreads in both directions. The downstream wave front travels away from the duct, but the upstream wave front—whose path in the (x,t)-plane is represented by a receding Mach line (Fig. 1)—travels into the duct and separates the disturbed region from the steady flow upstream. As long as this wave front does not meet a shock-wave, the velocity and state is continuous across it, and the coefficients in equations (11) and (12) may therefore be replaced by their local values in the steady solution. When this is done (use being made of (6) and (13)), and the two equations are added, the relation

$$D_-\sigma/Dt = \frac{\gamma+1}{4A}\sigma(\sigma+F(x))+G(x) \tag{14}$$

is obtained on the wave front, where σ stands for $D_+ s/Dt$, and

$$F(x) = -\frac{U'}{2(\gamma+1)A}[(\gamma+1)(U-A)^2-(9-3\gamma+4U/A)(U^2-A^2)]. \tag{15}$$

A particular solution of (14) is known, for on the upstream side of the wave front σ must equal its value in the steady solution,

$$\Sigma = D_+ S/Dt = (U+A)S' = -\tfrac{1}{2}A(1+U/A)^2U' \tag{16}$$

(by (7), (6), and (13)). By (14) the value of σ on the downstream side must therefore satisfy the equation

$$(4A/(\gamma+1))D_{-}(\sigma-\Sigma)/Dt = (\sigma-\Sigma)[\sigma+\Sigma+F]. \tag{17}$$

With $(\sigma-\Sigma) = 1/z$, this becomes

$$D_{-}z/Dt = -((\gamma+1)/(4A))[1+(2\Sigma+F)z], \tag{18}$$

and upon integrating,

$$\left.\begin{aligned} Kz &= K_0 z_0 - \frac{\gamma+1}{4}\int_0^t (K/A)\,dt \\ \text{where}\quad K/K_0 &= \exp\left\{\frac{\gamma+1}{4}\int_0^t \frac{2\Sigma+F}{A}\,dt\right\} \end{aligned}\right\}, \tag{19}$$

both integrals are taken along the path of the receding wave front in the (x,t)-plane, and values taken on this wave front at time $t = 0$ are distinguished by a suffix 0. Let $M = U/A$; then, by (15) and (16),

$$2\Sigma+F = -(3+1/M)[2+(\gamma-1)M^2]AU'/(\gamma+1), \tag{20}$$

which is bounded, and therefore z must vanish after a finite time when $z_0 > 0$, but z does not vanish when $z_0 < 0$.

Since

$$Du/Dt-DU/Dt = (\partial/\partial t+u\partial/\partial x)(u-U) = -\tfrac{1}{2}(\sigma-\Sigma) = -1/(2z) \tag{21}$$

on the receding wave front, by (7) to (10), (6), and (13), a zero of z implies an infinite acceleration and hence, a breakdown of the isentropic flow and the formation of a shock. A positive value of z corresponds to a sudden drop in the acceleration of the fluid particles as the wave front passes over them, by (21), and to a sudden rise of the local rate of change of pressure, since

$$\partial p/\partial t = \tfrac{1}{2}\rho(A-U)(\sigma-\Sigma) \tag{22}$$

on the receding wave front, by (1) to (4) and (7) to (10). Therefore, if a gas flows steadily through a converging-diverging duct, with sonic speed at the throat and subsonic speed in the diverging part, then *any disturbance at the exit that starts with a sudden rise of the local rate of change of pressure must lead to shock-formation after a finite time.*

To calculate the time t_l after which the limit point appears on the wave front, note that

$$K = M^{\frac{1}{2}}(1-M)^{-2},$$

since $dx/dt = U-A$ on the receding wave front and by (19), (20), and (13); and hence,

$$M^{\frac{1}{2}}z/(1-M)^2 = [M^{\frac{1}{2}}z/(1-M)^2]_{t=0}+\frac{\gamma+1}{4}\int_{x_0}^{x} M^{\frac{1}{2}}(1-M)^{-3}A^{-2}\,dx, \tag{23}$$

where x_0 denotes the position of the wave front at time $t = 0$. If x_l denotes the value for which the right-hand side of (23) vanishes,

$$t_l = \int_{x_0}^{x_l} \frac{dx}{U-A}.$$

The asymptotic behaviour of z is best inferred from (18). We may assume that $U' < 0$ in a diffuser and hence $(2\Sigma+F) > 0$, by (20). Therefore, z increases with time when $z < -1/(2\Sigma+F)$, and decreases with time when $z > -1/(2\Sigma+F)$, and asymptotically, that is, as the receding wave front approaches the throat, $(1/z) = \sigma-\Sigma$ approaches the asymptotic value of $-(2\Sigma+F)$ on the wave front. But $F \to 0$, by (15), and hence, whenever $\sigma \neq \Sigma$ initially, $\sigma \to -\Sigma$ asymptotically, so that the local pressure gradient, $\partial p/\partial x$, just downstream of the wave front, tends to the *opposite* of the local pressure gradient in the original steady flow, just upstream of the wave front, by (22).† This asymptotic solution can have physical significance only when the flow remains isentropic near the wave front. When $z_0 < 0$, i.e. when the disturbance starts with a sudden drop of the local rate of change of pressure, the acceleration remains finite on the wave front; but knowledge of the further development of the disturbance is required in order to exclude the possibility of a shock forming in the interior of the disturbed region and overtaking the wave front. The asymptotic solution shows, however, that the steady, shock-free diffuser-flow cannot be re-established even when the flow remains isentropic for all time in the neighbourhood of the wave front.

3.1. The same results apply to the upstream wave front of any disturbance generated at the entry of a converging-diverging duct (with the end of the disturbance marked by a discontinuity of the acceleration) when the initial flow in the converging part is steady, supersonic, and shock-free, with sonic speed at the throat. The Mach line in the (x, t)-plane that separates the wave from the undisturbed region upstream is in this case the receding Mach line passing through the entry of the diffuser at the time at which the disturbance ends (Fig. 2). Again, shock formation is certain when the fluid particles suffer a sudden drop of acceleration as the wave front passes over them; by (22), this means that the local rate of change of pressure is negative during the ultimate stage of the disturbance.

3.2. The development of receding wave fronts propagated into an entirely subsonic, or entirely supersonic steady, shock-free flow in a converging-diverging duct, where sonic speed is not reached in the throat, may be discussed similarly. Take first the case where the flow is subsonic,

† This result was first suggested by Kantrowitz (**2**).

and the disturbance is generated at the exit. Then the receding wave front travels through the throat into the converging part and leaves the duct through the entry, and this restricts the time during which the solution (19) has physical significance even if the flow remains isentropic in the neighbourhood of the wave front.

When $z_0 < 0$, i.e. when the disturbance starts with a drop of the local rate of change of pressure, the acceleration remains bounded on the wave front, by (19) and (21). The sense of variation of σ on the downstream

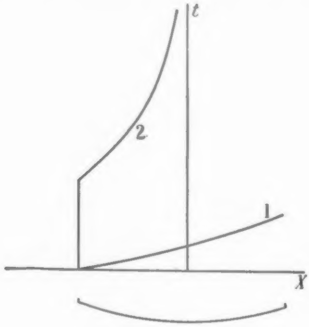

FIG. 2. 1. Advancing wave front. 2. Receding wave front in diffuser flow.

side of the wave front is towards the local value of $-(\Sigma+F)$, in the divergent part of the duct, by (18) and (20); in the converging part, where the steady flow is accelerated, and so $(2\Sigma+F) < 0$, it is towards Σ, that is, the wave front loses strength.

When $z_0 > 0$, there is a tendency to shock-formation in the diverging part; if this has not occurred when the wave front enters the converging part, the same tendency will persist, at least for some time (by (18) and (20), since $U' = 0$ at the throat). But later a different influence may prevail; to study this it is convenient to suppose that the duct extends indefinitely in the upstream direction.

Far upstream, U is small and UQ is approximately constant, and hence the integral in (23) remains bounded if $|x^{-2}Q| \to \infty$ as $x \to -\infty$. Therefore, even if the converging part extends indefinitely upstream, the acceleration will remain bounded and tend to zero ultimately (by (21) and (23)) and shock-formation may never occur, provided (i) the duct converges more rapidly than a cone, at a great distance upstream, and (ii) the initial value of z is sufficiently big, i.e. the initial rise in the local rate of change of pressure at the exit is sufficiently small.

Next, take the case where the original, steady flow is entirely supersonic, and the disturbance is generated at the entry and is of finite duration. Then the receding wave front separating the disturbed region from the steady flow upstream travels through the throat and finally passes out of the duct through the exit. The behaviour of z is similar to that in the previous case, but the tendency to shock-formation when $z_0 > 0$, is more difficult to overcome; when A is small, $QA^{2/(\gamma-1)}$ is approximately constant (by (13)), and $|x^{-4/(\gamma-1)}Q|$ must tend to infinity as $x \to \infty$, in order that the integral in (23) should remain bounded. For air with $\gamma = 7/5$ that means that the area of cross-section must tend to infinity more strongly than the tenth power of the distance from the throat.

4. Advancing wave fronts

Consider now a converging-diverging duct in which a steady shock-free flow has been set up, and assume that a disturbance is generated at the entry only, with its start marked by a discontinuity of the acceleration. The path of the wave front that separates the disturbed region from the undisturbed, steady flow downstream is represented in the (x,t)-plane by the advancing Mach line passing through the entry at the beginning of the disturbance (Fig. 2). Whatever type of steady flow has been set up to begin with—diffuser-flow, or nozzle-flow, or entirely subsonic, or entirely supersonic flow—this advancing wave front travels through the throat and ultimately passes out of the duct.

On the wave front, r, s, and x are again known as functions of time, and in the same way as equation (14) for σ, we find for the variation of $\lambda = D_- r/Dt$ on the wave front the equation

$$D_+ \lambda/Dt = ((\gamma+1)/(4A))\lambda(\lambda+P(x))+V(x).$$

A particular solution, $\Lambda = D_- R/Dt$, is again known, so $y = 1/(\lambda-\Lambda)$ satisfies the equation

$$D_+ y/Dt = -((\gamma+1)/(4A))[1+(2\Lambda+P)y], \tag{24}$$

and a short calculation shows that

$$2\Lambda+P = (1/M-3)[2+(\gamma-1)M^2]AU'/(\gamma+1). \tag{25}$$

The solution of (24) is

$$\left.\begin{aligned} Ly &= L_0 y_0 - \frac{\gamma+1}{4}\int_0^t (L/A)\,dt \\ \text{where}\qquad L/L_0 &= \exp\left\{\frac{\gamma+1}{4}\int_0^t \frac{2\Lambda+P}{A}\,dt\right\} \end{aligned}\right\}, \tag{26}$$

and both integrals are taken along the path of the advancing wave front in the (x, t)-plane. Therefore, $(U+A)\,dt = dx$, and by (25) and (13),

$$L = M^{\frac{1}{2}}(1+M)^{-2},$$

and, by (26),

$$M^{\frac{1}{2}}y/(1+M)^2 = [M^{\frac{1}{2}}y/(1+M)^2]_{t=0} - \frac{\gamma+1}{4}\int\limits_{x_0}^{x} M^{\frac{1}{2}}(1+M)^{-3}A^{-2}\,dx. \quad (27)$$

This equation shows that there is a definite *tendency to shock-formation* when $y_0 > 0$ in *both diffuser-flow and nozzle-flow* and also all other steady flows. Since

$$Du/Dt - DU/Dt = \tfrac{1}{2}(\lambda-\Lambda) = 1/(2y) \quad (28)$$

and

$$\partial p/\partial t = \tfrac{1}{2}\rho(U+A)(\lambda-\Lambda) \quad (29)$$

on an advancing wave front, by (1) to (4), (6), (7) to (10), and (13), $y > 0$ means a sudden rise in the acceleration of the fluid particle, as the wave front passes over it, and $y_0 > 0$ for a disturbance starting with a rise in pressure. Note, however, that the wave front may have passed the exit of the duct before the acceleration has become large on it; and even if a shock forms at the front of the wave while the front is still inside the duct, the shock travels with an absolute speed greater than the local value of $(U+A)$ just downstream of the shock and passes the exit after a very short time.

Moreover, even if the duct extends indefinitely in the downstream direction, the tendency to shock formation may ultimately be overcome if (i) the integral in (27) converges as $x \to \infty$, and (ii) the initial rise in the local rate of change of pressure is sufficiently small. For a duct with steady subsonic flow in the divergent part, this integral converges if the duct diverges ultimately more rapidly than a cone† (cf. section 3.2 above). For a duct with steady, supersonic flow in the divergent part, the integral converges only if $|x^{-4/(\gamma-1)}Q| \to \infty$ as $x \to \infty$. If the tendency to shock-formation is indeed overcome, $(\lambda-\Lambda)$ tends to zero asymptotically in all ducts.

When $y_0 < 0$, on the other hand, y remains negative, and the acceleration remains finite on the wave front. Equations (24) and (25) show that, when $M > \frac{1}{3}$, $|\lambda-\Lambda|$ decreases with time in a nozzle; but in a diffuser the sense of variation of λ is towards the local value of $-\Lambda-P$, so long as $M > \frac{1}{3}$. For any other duct, the tendency in the converging part is the same as in a nozzle or a diffuser according to the sign of U'; and similarly for the diverging part.

† Boundary layer separation may prevent this.

5. Conclusions

In order to summarize the results obtained for the development of wave fronts travelling into a steady, shock-free flow in a duct of slowly varying cross-section, it is convenient to classify the wave fronts in two different ways, namely (i) as to whether they are connected with an expansion or with a compression, and (ii) as to whether they travel through the duct and pass out of it, or whether they travel towards a sonic throat so as to approach it asymptotically.

All compression wave fronts, i.e. all wave fronts raising the local rate of change of pressure when passing through a section of the duct, show at least an initial tendency to shock formation. All expansion wave fronts show a tendency opposed to shock formation. In fact, when they travel into a steady accelerating flow they show a permanent tendency to decay. But when they travel into steady decelerating flow they show at least an initial tendency to establish the local pressure gradient appropriate to the accelerating steady flow solution, i.e. the opposite of the local pressure gradient of the original steady flow.

Wave fronts are called 'advancing' or 'receding' according to whether they travel downstream or upstream, with respect to the moving fluid. Advancing wave fronts generated at the entry of a duct travel through the duct and pass out of the exit after a short time; advancing wave fronts generated at the exit do not enter the duct at all. Receding wave fronts generated at the entry travel into the duct if the steady flow is supersonic near the entry, and penetrate as far as it remains supersonic. Receding wave fronts generated at the exit travel into the duct if the flow is subsonic, and penetrate as far as it remains so. If there is a throat where the steady flow is sonic, then all receding wave fronts entering the duct approach the (nearest) sonic throat asymptotically.

The tendencies of development found for compression and expansion waves are not only initial, but permanent, for all wave fronts approaching asymptotically a sonic throat. For instance, if a steady shock-free diffuser flow could be set up, then any disturbance generated subsequently at the exit so as to start with a rise of the local rate of change of pressure must inevitably lead to the formation of a shock after a finite time (section 3).

For wave fronts travelling right through a duct, conditions are complicated by the fact that a new tendency makes itself felt as soon as the wave front has passed through a throat and this may counteract the initial tendency. To clarify this question it is convenient to suppose that the duct extends indefinitely in the direction in which the wave front is travelling. It is then found that the initial tendency must prevail if, asymptotically, the duct does not diverge more strongly than a cone, in

the case of a steady subsonic flow, or if the cross-section does not tend to infinity more strongly than the tenth power of the distance from the throat (for air), in the case of a steady supersonic flow (section 3.2). Otherwise, a tendency for the wave front to decay must ultimately prevail, provided the wave front has not been overtaken by a shock. For instance, suppose a steady nozzle flow leading to a very low pressure has been set up in a duct the supersonic portion of which diverges more rapidly than the tenth power (for air) of the distance from the throat. Then there exists a limiting initial strength for any compression wave front (not, in general, the same for different ducts); if it is stronger, a shock must form before the tendency to decay can prevail over the earlier tendency to steepen; if it is weaker, no shock is formed in the neighbourhood of the wave front and, asymptotically, the strength tends to zero (section 3.2).

These results are seen to explain the difference in instability between steady nozzle flows and steady diffuser flows. From a practical point of view, the contrast between the two types of flow is due, in the first place, not to the 'accumulation' of disturbances, which is a property of compression waves, but to the fact that in nozzle flows all wave fronts, and the shocks to which they may lead, are blown out of the duct immediately. In diffuser flows, on the other hand, all receding wave fronts, and for sufficiently small disturbances also their shocks (**4**), are caught in the duct.

The distinction is accentuated by the tendency even of expansion wave fronts not to decay when travelling into a decelerating, steady flow.

From a mathematical point of view, it is of interest to note that the present stability investigation for a non-linear partial differential equation leads to results resembling those of stability investigations for non-linear, ordinary differential equations leading to 'limit cycles'.† If attention is confined entirely to the equations of isentropic flow, the pressure gradient just downstream of *any* receding wave front with discontinuous acceleration that is caught in a duct is seen to tend asymptotically to the local pressure gradient of the steady accelerating flow. This steady solution shows some analogy to a 'stable limit cycle'; the steady, decelerating flow, on the other hand, shows the same analogy to an 'unstable limit cycle'.

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† The author is indebted to Dr. F. G. Friedlander for pointing out that the Riccati equation plays an important role also for these equations.

APPENDIX

(i) Brown (**1**) studies wave fronts of the same type as those treated above. He introduces characteristic coordinates ξ, η, where ξ is constant on receding Mach lines, and η on advancing Mach lines, so that the equations (7) and (8) may be replaced by

$$\partial x/\partial\xi-(u+a)\,\partial t/\partial\xi = 0, \qquad \partial r/\partial\xi+\tfrac{1}{2}auH\,\partial t/\partial\xi = 0, \tag{A 1}$$

$$\partial x/\partial\eta-(u-a)\,\partial t/\partial\eta = 0, \qquad \partial s/\partial\eta+\tfrac{1}{2}auH\,\partial t/\partial\eta = 0 \tag{A 2}$$

(equations 5.5 and 5.8 of (**1**); the definition of s given in (**1**) differs from the one adopted here by the sign). To calculate the steepening of a receding wave front, Brown assumes that r, s, $\partial r/\partial\eta$, $\partial s/\partial\eta$, x, t, $\partial x/\partial\xi$, $\partial t/\partial\xi$, $\partial x/\partial\eta$, $\partial t/\partial\eta$, $\partial^2 x/\partial\xi\partial\eta$, and $\partial^2 t/\partial\xi\partial\eta$ are continuous, but that $\partial s/\partial\xi$, at least, is discontinuous across a certain receding Mach line $\xi = \xi^*$.

Elimination of x from the first pair of the equations (A 1) and (A 2) gives

$$2a\frac{\partial^2 t}{\partial\xi\partial\eta}+\frac{\partial t}{\partial\xi}\frac{\partial}{\partial\eta}(u+a)-\frac{\partial t}{\partial\eta}\frac{\partial}{\partial\xi}(u-a)$$

$$= 2a\frac{\partial^2 t}{\partial\xi\partial\eta}+\left(\frac{3-\gamma}{2}auH\frac{\partial t}{\partial\eta}+\frac{\gamma+1}{2}\frac{\partial r}{\partial\eta}\right)\frac{\partial t}{\partial\xi}+\frac{\gamma+1}{2}\frac{\partial t}{\partial\eta}\frac{\partial s}{\partial\xi} = 0$$

(use is made of (3), (A 1) and (A 2)), which shows that the above assumptions are incompatible. In fact, a discontinuity of $D_+ s/Dt$ implies a discontinuity of both $\partial s/\partial\xi$ and $\partial t/\partial\xi$ (and hence, also of $\partial r/\partial\xi$ and $\partial x/\partial\xi$).

(ii) Kantrowitz (**2**) studies the whole of a pulse superimposed on the flow of the steady solution in the subsonic part of a diffuser, neglecting terms of the order of $(M-1)$ against unity, where M is the local Mach number of the steady solution. With regard to such an approximation it should be noted that, if the approximation is introduced into an equation like (7), or (8), or (11) and (12), and such an equation is then integrated, it is difficult to assess the asymptotic behaviour of the resulting expression without further investigation of the effect of the small terms over a very long time. In fact, on a wave front of the type studied in section **3** above

$$\int^{\infty} (U-A)\,dt = \int^{0} dx = O(1),$$

by (8), if the wave front does not meet a shock wave. As it stands, the argument of (**2**) is not certain to lead to a correct description of the asymptotic state of the pulse.

ON WAVES OF FINITE AMPLITUDE IN DUCTS

PART II. WAVES OF MODERATE AMPLITUDE

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SUMMARY

The results of Part I are extended to a first-order theory of advancing and receding waves travelling into steady, shock-free flow of inviscid, perfect gas in a duct of slowly varying cross-section. This includes, as special cases, spherical waves and waves with cylindrical symmetry.

The theory applies (i) to the neighbourhood of any wave front travelling into steady, shock-free flow, and (ii) to the waves caused by any disturbance of finite duration and extent such that the velocity and pressure differ little from their respective local values in the steady flow (section IV. 2). No assumption is implied that the local rates of change of velocity and pressure are small, and shock waves may form. Only a partial solution, however, is given for the interaction of an advancing wave with a receding wave of comparable strength (section IV. 1).

In a *first approximation*, the fluid acceleration, the local rate of change of pressure, and the limit lines are found (sections III. 2, III. 4). A *second approximation* gives the velocity and the pressure (sections IV, IV. 1) and the shock path, to the first order in the shock strength (sections V, V. 1).

I. Introduction

RIEMANN (1) investigated the one-dimensional, unsteady flow of an inviscid gas and found that any disturbance of finite extent and duration set up in an originally steady flow generates two waves, one advancing and the other receding with respect to the moving fluid, which are propagated into the gas not initially affected by the disturbance. After a certain period of interaction (depending on the way in which the disturbance is set up), these two waves will become separated from each other, and each will then be a 'simple wave'. Between the two waves the flow will again be steady.

Conditions are similar in the unsteady flow of inviscid gas in a duct, the area of cross-section of which changes slowly enough for the velocity and pressure to depend primarily on two independent variables only, viz. the position, x, and the time, t. If a disturbance† of finite extent and duration is set up in an originally steady, shock-free flow, it generates two primary waves, one advancing and the other receding, which become separated

† It is assumed that a shock wave is not formed immediately. That case, however, can be treated by the method described below, provided the shock is weak (cf. 2).

after a certain period of interaction. These waves are not simple waves. Owing to the variation of the area of cross-section, the waves are continually distorted, as they travel into gas originally in steady flow, and the process of distortion is accompanied by one of refraction. The receding wave thus generates an advancing wave by refraction, and the advancing wave generates a receding wave. The refracted wave interacts with the primary wave, gets itself distorted as it travels through the duct, and by the accompanying process of refraction contributes to the primary wave. The primary wave passes any station in the duct within a finite time, but it leaves behind it the refracted wave as a 'tail', and the flow between the two primary waves is not steady, but rather a superposition on a steady flow of the interaction of the two tails accompanied by the effects of distortion and refraction of each of them.

The original steady flow does not give rise to any wave, and since the process of refraction is a continuous one, the strength of the refracted wave is built up only gradually with distance from the front of the primary wave travelling into steady flow. This makes it possible to treat certain types of transient waves by successive approximations. A detailed discussion of the type of wave to which the theory applies is given in section IV. 2.

It is found convenient to take as the fundamental dependent variables not the velocity and pressure but certain characteristic derivatives of them which are not determined directly by the equations of motion (or rather the differences between these derivatives and their local values in the steady flow). Each primary wave is characterized by the distribution of one such derivative; the distribution of the other derivative characterizes the refracted wave. In what follows, the successive approximations are carried out in detail for the case of a receding wave generated by a disturbance at the exit of a duct. For other cases the results are quoted. For a receding wave, the 'primary' derivative is

$$\Delta\sigma \equiv [\partial/\partial t+(u+a)\,\partial/\partial x]\{(a-A)/(\gamma-1)-\tfrac{1}{2}(u-U)\},$$

where u is the velocity, a the local speed of sound, and U and A are their respective local values in the steady flow. (See list of symbols at the end of section II.)

For a receding wave, the *first approximation* consists in the calculation of the distribution of $\Delta\sigma$ (equation 35), which results from the combined effect of the growth typical for a simple wave and the distortion due to the change in the area of cross-section. This approximation yields also the local rate of convergence of the receding Mach lines (equation 39), and the receding limit lines (equation 40), but the deviations of velocity and pressure from their local values in the steady flow are neglected.

Once the distribution of $\Delta\sigma$ is known to the first order, it is found that, as a *second approximation*, these deviations of velocity and pressure can be obtained to the first order (section IV). The value of $\Delta\sigma$ at any one point in the wave is found to depend only on the value of $\Delta\sigma$ in the disturbance at the exit at a certain instant of time, but the deviations of velocity and pressure depend also on all the earlier values of $\Delta\sigma$ at the exit. The *second approximation* yields also the shock path, to the first order in the shock strength (sections V, V. 1). Explicit formulae for the shock path are only given for shocks starting at, or near, the wave front, in which case certain effects of distortion due to the change in the area of cross-section can be neglected.

The refracted wave is found to be only a second-order effect, i.e. small even compared with the velocity deviation (sections III. 5, IV. 2) and it is neglected hereafter, together with the effects of its distortion, its interaction with the primary wave, and the higher-order effect of secondary refraction due to the distortion of the refracted wave.

The *second approximation* can also be extended to give the velocity correction for a region of interaction of two primary waves of comparable strength (section IV. 1).

The feature distinguishing the present theory from an acoustic theory of 'waves of small amplitude' is that values of any magnitude of the fluid acceleration and of the local rate of change of pressure are admitted. The feature distinguishing it from a full theory of 'waves of finite amplitude', like that of Riemann (**1**), is that the velocity and pressure themselves are only permitted to differ little from their local values in the steady flow.

The present theory is similar in many respects to the first-order theory of the steady, supersonic, axially symmetrical flow past slender bodies of revolution developed by Whitham (**2**). Both theories are based on the same physical ideas, but the mathematical approach is rather different. Whitham's theory is developed from a linearized theory of a nearly uniform flow, whereas in the problem studied here not even the basic flow is uniform and the theory is developed directly from the non-linear equations of motion. On the other hand, a fundamental feature of Whitham's theory is the correct description of the asymptotic behaviour of the primary wave, whereas the present theory applies only to waves passing out of the duct after a finite time; this is sufficient, since only steady, shock-free flows are stable for which all waves have this property.

Note. The notation follows that used in Part I, and *the equations are numbered so that numbers up to, and including,* (29) *refer to equations of Part I.* For convenience, a list of symbols is given at the end of section II.

II. Remark on the boundary conditions

Consider the strictly one-dimensional flow of inviscid gas. A well-known problem is that of the flow in a long pipe containing a piston and, on both sides of it, gas initially at rest and in thermodynamic equilibrium; the piston is set in motion with finite acceleration from the time $t = 0$ onwards. Then two waves are propagated into the gas, one from each side of the piston, and each is a simple wave, since it travels into gas at rest. In each of the simple waves one of the two Riemann variables (cf. equation 4) is constant, but it is not the same Riemann variable that is constant in both waves, one of which is 'advancing', while the other is 'receding'. In particular, a Riemann variable is constant on each side of the piston, but it is not the same Riemann variable that is constant on both sides, although both surfaces of the piston have the same velocity.

This simple set-up depends essentially on the assumption that the gas is initially at rest. If the gas is initially flowing through the pipe with uniform, subsonic speed, we may, for instance, imagine a disturbance set up in the following way. A segment of the pipe wall is made of porous material and suction is applied from $t = 0$ onwards. (Suppose that a useful result can be obtained even when the ensuing deviations from one-dimensional flow are neglected—it is not the purpose of this investigation to study whether, and how, a disturbance could be set up that leaves the flow one-dimensional.) From each border cross-section of the porous segment of the pipe a simple wave will be propagated into the gas, but the flow inside the porous segment will be of a more complex type. (That the waves are simple waves is easily seen as follows. In the (x, t)-plane, every point outside the strip representing the porous segment is connected by a Mach line not passing through the strip to a region representing the initial, uniform flow. On such a Mach line, one Riemann variable is constant. In a region of uniform flow, both Riemann variables are constant.)

It is not the purpose of this investigation to study the flow inside the pipe segment where the disturbance is set up, and it is therefore convenient to suppose that the porous segment is replaced by a narrow slit in the wall of the pipe. For mathematical convenience, let the width of the slit tend to zero. The slit is then represented in the (x, t)-plane by a line, or at least segment of a line, $x =$ constant across which both Riemann variables are discontinuous. On each side of the line, one Riemann variable is constant, but it is not the same Riemann variable on both sides. On each side of the line, the distribution of the other Riemann variable can be prescribed arbitrarily, and it must be prescribed in order to determine the solution. Alternatively, the distribution of its time derivative, or of the primary derivative, may be prescribed.

Conditions are very similar in a duct of slowly varying cross-section in which a disturbance is set up in an initially steady, isentropic, subsonic flow. A wave is propagated into the steady flow on each side of the position where the disturbance is set up. It can be shown that, as one proceeds along a Mach line from the region of steady flow in the (x, t)-plane into the region of unsteady flow, for a short distance, the rate of change of one of the Riemann variables (primary derivative) is much larger than the rates of change of the velocity and the pressure, respectively, while the rate of change of the other Riemann variable is much smaller even than those of the velocity and pressure.† The latter Riemann variable is the one that would be constant on the Mach line in strictly one-dimensional flow.

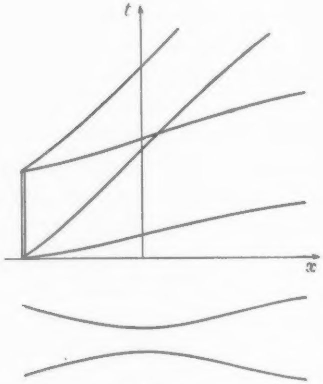

FIG. 1.

It is convenient to study separately the two waves generated by the disturbance. In what follows, the disturbances will therefore be represented by a line $x =$ constant in the (x, t)-plane, or a segment of such a line, on which the distribution of the primary derivative is prescribed. It is assumed to be bounded, but not necessarily continuous. This is found to imply that the velocity and the pressure are continuous on the line.

When the initial steady flow is supersonic, the situation is complicated by the fact that both waves travel downstream with respect to an observer at rest. The region in the (x, t)-plane just downstream of the line representing the disturbance is therefore a region of interaction of the two waves (Fig. 1). However, if the disturbance is of finite duration the waves separate eventually. The results mentioned above regarding the orders of magnitude of the respective rates of change hold near the wave front advancing into steady flow and near the wave front receding into steady flow. Similarly the theory described below applies to each of the waves, after their emergence from the region of interaction, but for the region of interaction itself only a partial solution is obtained here. For the formulae given below to apply without modification, it is necessary to prescribe, as a boundary condition, the distribution of the primary derivative on a line $x =$ constant outside the region of interaction in the (x, t)-plane.

† Actually it is not these four variables themselves to which that statement applies, but the difference between the variables and their respective local values in the steady flow.

List of symbols

$x =$ distance along duct, measured downstream.

$t =$ time, measured from beginning of disturbance.

$u =$ velocity.

$a =$ speed of sound.

$r = a/(\gamma-1)+\frac{1}{2}u$.

$s = a/(\gamma-1)-\frac{1}{2}u$.

$p =$ pressure.

$\gamma =$ ratio of specific heats.

$U, A, P, R, S =$ local values of u, a, p, r, s, respectively, in steady flow solution.

$\Delta =$ difference between local values in unsteady and in steady flow, e.g. $\Delta u = u-U$.

$M = U/A$.

$\rho_+ = (u+a)/(U+A)$.

$\rho_- = (u-a)/(U-A)$.

$H = (1/Q)\, dQ/dx$, where Q denotes area of cross-section.

$v = \dfrac{1}{2}auH$.

$V = \dfrac{1}{2}AUH$.

$D_+/Dt = \partial/\partial t+(u+a)\partial/\partial x$.

$D_-/Dt = \partial/\partial t+(u-a)\,\partial/\partial x$.

$\sigma = D_+ s/Dt$. $\qquad \lambda = D_- r/Dt$.

$\Sigma = D_+ S/Dt$. $\qquad \Lambda = D_- R/Dt$.

$\Delta\sigma = \sigma-\Sigma$. $\qquad \Delta\lambda = \lambda-\Lambda$.

$z = 1/\Delta\sigma$. $\qquad y = 1/\Delta\lambda$.

$K = M^{\frac{1}{2}}(M-1)^{-2}$. $\qquad L = M^{\frac{1}{2}}(M+1)^{-2}$.

$I = \dfrac{\gamma+1}{4}\displaystyle\int_{x_0}^{x}\frac{K\,dx}{A(A-U)}$. $\qquad J = \dfrac{\gamma+1}{4}\displaystyle\int_{x_0}^{x}\frac{L\,dx}{A(A+U)}$.

$d\tau$, $d\tau'$ defined in section III. 3.

Suffix 0 denotes values taken at exit of duct, for wave receding into steady, subsonic flow.

l denotes values taken at limit point.

s denotes values taken at shock point.

1 denotes values taken at shock point, on upstream side of shock.

2 denotes values taken at shock point, on downstream side of shock.

III. The first approximation

Consider the receding wave generated by a disturbance at the exit of a duct in which a steady shock-free flow has been set up for $t \leqslant 0$, with subsonic flow near the exit. Assume that the disturbance is such that

$$\Delta s_0/S_0 = O(\epsilon) \quad (\epsilon \ll 1), \tag{30}$$†

at the exit, for $0 \leqslant t \leqslant T_0$, where T_0 may be the time during which the disturbance lasts.

If s_0 is given for $0 \leqslant t \leqslant T_0$, the flow is determined, by the general Uniqueness Theorem (3), between the wave front receding into the steady flow and the receding Mach line passing the exit at time T_0. Therefore, if it is possible to find a consistent, approximate solution on the basis of the additional assumption that

$$\Delta u/U = O(\epsilon), \qquad \Delta a/A = O(\epsilon), \qquad \Delta\lambda/\Lambda = O(\epsilon) \quad (\epsilon \ll 1), \tag{31}$$

everywhere in the region where the flow is so determined (and this will be done in what follows), it may be concluded that the assumption (31) is justified if the assumption (30) is accepted.

III. 1. Equations (4) and (6) of Part I remain valid in a region of unsteady flow when capital letters are substituted throughout, but not equations (9) and (10); in fact,

$$D_+R/Dt = -\rho_+V, \qquad D_-S/Dt = -\rho_-V, \tag{32}$$

by (7), (8), (4), and (13). An equation for the variation of $\sigma = D_+s/Dt$ on a receding Mach line can be established in a way similar to that given in sections 2 and 3 of Part I; after a short calculation involving use of (6), (7), (9), (10), (12), and (32), it is found to be

$$\begin{aligned}\frac{D_-}{Dt}(\Delta\sigma) = {} & \frac{\gamma+1}{4a}(\Delta\sigma - v + \rho_-V)[\Delta\sigma + 2\Sigma + F + \Delta\lambda + \Gamma] + \\ & + \frac{\gamma-1}{2}H[vr - \rho_+\rho_-VR + (s-\rho_-S)\Sigma] - \\ & - \frac{V}{U-A}\left[\frac{\gamma+1}{2}(1-\rho_-)\Sigma + \frac{3-\gamma}{2}(v - \rho_+\rho_-V)\right] - \\ & - (v-\rho_-V)\left[(u+a)\frac{d}{dx}\log H - \frac{\gamma-1}{2}asH + \frac{\gamma+1}{2}\frac{V}{U-A}\right],\end{aligned} \tag{33}$$

where F is defined by (15) and

$$\begin{aligned}\Gamma = {} & \frac{2}{\gamma+1}[(3-\gamma)(u-U) + (\gamma-1)H\{(a-A)s + (s-S)A\}] - \\ & - V\left[\frac{u-a-U+A}{U+A} + \frac{u+3a-U-3A}{U-A}\right].\end{aligned}$$

On a wave front receding into steady flow, (33) is seen to reduce to (17).

† See note at the end of section I.

The variation of $\lambda = D_-r/Dt$ on an advancing Mach line is governed by a similar equation, which reduces to (24) on a wave front advancing into steady flow.

III. 2. By (31), $\Delta s/S$, $\Delta r/R$, $\Delta v/V$, ρ_+-1, and ρ_--1 are all $O(\epsilon)$ and (33) becomes

$$\frac{D_-}{Dt}(\Delta\sigma) = \frac{\gamma+1}{4a}(\Delta\sigma)\{\Delta\sigma+2\Sigma+F+O(\epsilon)\}+O(\epsilon).$$

With $z = 1/\Delta\sigma$, this becomes

$$\frac{D_-z}{Dt} = -\frac{\gamma+1}{4A}\{[2\Sigma+F+O(\epsilon)]z+1+(1+\Delta\sigma)O(\epsilon)\}+\Delta\sigma^2O(\epsilon). \tag{34}$$

Therefore, if $\Delta\sigma$ is $O(1)$, z satisfies, to the first order, equation (18) of Part I. The solution is given by equation (23):

$$K(x)z(x,t) = K_0 z_0+I(x), \tag{35}$$

where

$$K(x) = M^{\frac{1}{2}}(1-M)^{-2},$$

$$I(x) = \frac{\gamma+1}{4}\int\limits_{x_0}^{x}\frac{K\,dx}{A(A-U)},$$

and a suffix 0 denotes values at some reference point on the receding Mach line through the point (x,t). In the following the reference point will be taken to be the point (x_0,t_0) where this Mach line passes the exit and where z_0 is assumed to be prescribed (cf. section II).

On the other hand, if $\Delta\sigma$ is itself only $O(\epsilon)$, then (34) becomes, to the first order,

$$D_-z/Dt = -(\gamma+1)[2\Sigma+F]z/(4A),$$

and so

$$Kz = \text{constant on a receding Mach line}, \tag{35a}$$

which is the special case of (35) obtained when $K_0 z_0 \gg I$.

Finally, if $\Delta\sigma$ is large, (33) becomes

$$D_-(\Delta\sigma)/Dt = (\gamma+1)(\Delta\sigma)^2/(4a),$$

to the first order. It will be shown in the following section that the occurrence of a singularity of $\Delta\sigma$ does not imply a breakdown of the assumption that $\Delta a/A = O(\epsilon)$, and hence, to the first order,

$$D_-z/Dt = -(\gamma+1)/(4A),$$

when z is small, and the solution is, again, (35).

The fluid acceleration and the local rate of change of pressure are given by (21) and (22), respectively, to the first order.

III. 3. It is useful to consider briefly the differential geometry of the Mach line field. Let $d\tau$ be the time interval between two neighbouring receding Mach lines, measured along an advancing Mach line; and $d\tau'$ the

time interval between two neighbouring advancing Mach lines, measured along a receding Mach line. Consider two points, Q and Q', at a small distance apart in the (x, t)-plane (Fig. 2; this figure has been chosen to represent the case of supersonic flow). Approximately,

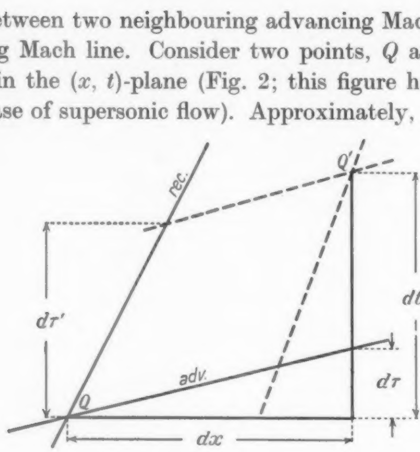

FIG. 2.

$$dx = x(Q')-x(Q)$$
$$= (u-a)\,d\tau'+[u+a+\{D_-(u+a)/Dt\}\,d\tau'][d\tau+\{D_-(d\tau)/Dt\}\,d\tau']$$
$$= (u+a)\,d\tau+[u-a+\{D_+(u-a)/Dt\}\,d\tau][d\tau'+\{D_+(d\tau')/Dt\}\,d\tau],$$

and
$$dt = t(Q')-t(Q) = d\tau+d\tau'+\{D_+(d\tau')/Dt\}\,d\tau$$
$$= d\tau'+d\tau+\{D_-(d\tau)/Dt\}\,d\tau',$$

whence to the first order,

$$(2a/d\tau)D_-(d\tau)/Dt = D_+(u-a)/Dt-D_-(u+a)/Dt. \tag{36}$$

When $\Delta\sigma$ (but not $\Delta\lambda$) is large, this becomes, to the first order,

$$\frac{1}{d\tau}\frac{D_-(d\tau)}{Dt} = -\frac{\gamma+1}{4a}\Delta\sigma = -\frac{1}{\Delta\sigma}\frac{D_-}{Dt}\Delta\sigma,$$

i.e.
$$D_-(\Delta\sigma\,d\tau)/Dt = 0, \tag{37}$$

by (6), (9), (10), and (33), since u and a are necessarily bounded. If $\Delta\sigma$ is replaced by σ, equation (37) characterizes a simple, receding wave in the one-dimensional, unsteady flow of an inviscid gas, and a point where $1/\sigma = 0$ is called a limit point.† The same definition will be adopted for the flow studied here, and since $1/\Sigma \neq 0$, (37) shows that the variation of σ near a limit point is, to the first order, independent of the variation of the cross-section of the duct. Singularities of the limit type in one-dimensional, unsteady flow, and in the analogous problem of steady, two-dimensional, supersonic flow, have been studied by Craggs (**4**), Meyer (**5, 6**), and Stocker (**6**), and extensive use will be made in sections V and V.1 of the qualitative results obtained by these authors.

† Strictly speaking, a 'receding' limit point. A point where $1/\lambda = 0$ is an 'advancing' limit point, but such a limit point does not occur in a receding wave of the type studied here.

A singularity of σ implies an infinite gradient of a in the advancing Mach direction (cf. list of symbols), just as in a simple receding wave. But since λ is bounded, there is no comparably strong gradient of a in the receding Mach direction (in a simple receding wave, $\lambda \equiv 0$ and a is constant on receding Mach lines); and hence, if a is small on a receding Mach line, at a distance from a limit line where σ is still $O(1)$, then a will be small of the same order on this Mach line, near the limit point.

Equation (36) may be integrated, to the first order. By (6), (7), (8), and (31),

$$\begin{aligned} D_+(u-a)/Dt &= D_+(U-A)/Dt-\tfrac{1}{2}(\gamma+1)\Delta\sigma+O(\epsilon) \\ &= -\tfrac{1}{2}(\gamma+1)\Delta\sigma+[(U+A)/(U-A)]D_-(U-A)/Dt+O(\epsilon), \end{aligned}$$

and by (6), (10), (31), and (32),

$$D_-(u+a)/Dt-D_-(U+A)/Dt = O(\epsilon),$$

whence (36) becomes

$$\frac{1}{d\tau}\frac{D_-}{Dt}(d\tau) = -\frac{\gamma+1}{4A}\Delta\sigma+\frac{1}{2A}\left\{\frac{U+A}{U-A}\frac{D_-}{Dt}(U-A)-\frac{D_-}{Dt}(U+A)\right\}+O(\epsilon). \tag{38}$$

But $$\frac{1}{2A}\left\{\frac{U+A}{U-A}\frac{D_-}{Dt}(U-A)-\frac{D_-}{Dt}(U+A)\right\} = \frac{1}{M-1}\frac{D_-M}{Dt},$$

and by (35) and (8),

$$D_-(Kz)/Dt = (U-A)\,dI/dx = -(\gamma+1)K/(4A),$$

and so from (38), to the first order,

$$[(M-1)Kz]^{-1}\,d\tau = \text{constant on receding Mach lines.} \tag{39}$$

III. 4. The direct implications of equation (35) have been discussed in detail in sections 3 and 3.1 of Part I. It may be worth repeating that a limit point ($z = 0$) occurs upstream of the exit on all the receding Mach lines on which z is positive at the exit, and only on those. The position, x_l, of the limit point is given by

$$I(x_l)+K_0 z_0 = 0. \tag{40}$$

Thus, to each interval of time during which z_0 is positive there corresponds a limit line, and the limit point on it nearest to the exit occurs on the Mach line on which z_0 takes its least value, i.e. on which σ takes its greatest value at the exit. It is also the limit point occurring at the earliest time, for the slope of the limit line is $dt/dx = 1/(u-a) \neq 0$ and it is a continuous line if z_0 is continuous.† This limit point is of particular interest since the Rankine–Hugoniot equations for a weak shock suggest that it starts at the first limit point to occur in the flow (**5**).

† If z_0 is not continuous, edges of regression intervene (cf. **6**), but the statement concerning the limit point occurring at the earliest time remains true.

Equation (35) applies also to receding waves travelling into steady, supersonic flow in the convergent part of a duct. For the choice of the reference points in that case, cf. section II.

This completes the proof of Part I of the instability of all steady shock-free flows which attain sonic velocity in a throat but are not nozzle flows. For (31) holds for arbitrary ϵ in a sufficiently narrow strip of receding Mach lines adjacent to any wave front receding into steady flow, since Δu, Δa, and $\Delta\lambda$ vanish on such a wave front. Therefore, only steady flows for which all wave fronts pass out of the duct after a finite time will be considered hereafter.

The *first approximation* to the flow in a receding wave where the assumptions (31) hold may therefore be summed up as follows. The velocity and the speed of sound do not differ from their local values in the steady flow; $\Delta\lambda = D_{-}(r-R)/Dt$ is zero, but $\Delta\sigma = D_{+}(s-S)/Dt$ is given by (35), where $z = 1/\Delta\sigma$. A limit line may occur and its approximate position is given by (40), in spite of the fact that this *approximation* does not distinguish between the Mach lines of the disturbed flow and of the steady flow. In particular, the approximate position and time of the first limit point is found, which may be interpreted as the position and time at which a shock wave starts.

Similar results are obtained for an advancing wave where

$$\Delta s/S = O(\epsilon), \qquad \Delta r/R = O(\epsilon), \qquad \Delta\sigma/\Sigma = O(\epsilon) \quad (\epsilon \ll 1). \tag{41}$$

In that case $\Delta\sigma$ is zero, in the *first approximation*, and $\Delta\lambda = 1/y$ is given by

$$\left.\begin{aligned} L(x)y(x,t) &= L_0 y_0 - J(x), \\ \text{where} \qquad L(x) &= M^{\frac{1}{2}}/(1+M)^{-2}, \\ J(x) &= \frac{\gamma+1}{4}\int_{x_0}^{x} \frac{L\,dx}{A(A+U)} \end{aligned}\right\} \tag{42}$$

and a suffix 0 denotes values taken at some reference point on the advancing Mach line through the point (x, t); for the choice of the reference point, cf. section II. The fluid acceleration and the local rate of change of pressure are given by (28) and (29), respectively, to the first order.

III. 5. From a mathematical point of view, the assumption made in (31) that $\Delta\lambda/\Lambda$ is $O(\epsilon)$, at most, is not altogether essential (cf. section IV. 1). It is important, however, for the physical understanding of the problem to note that $\Delta\lambda$ is indeed small. To see this, consider first equation (35a). In a duct of constant cross-section, a zero of σ at the exit would imply a receding branch line (6), which is not to be confused with the advancing Mach lines, all of which would be branch lines since the receding wave would be a simple wave. In a duct of varying cross-section, a zero of $\Delta\sigma$

at the exit implies at least a narrow strip of receding Mach lines on which (35a) holds. Such a strip might be called an 'acoustic' strip, for in such a strip, and only in it, the present theory reproduces the predictions of Acoustic, or Linearized, theory.

Now consider the advancing Mach lines. An equation for $\Delta\lambda$ holds on them which is analogous to equation (33) for $\Delta\sigma$, and the solution is given by (42), except that $L(x)$ has to be replaced by

$$L' = L(x)\exp\left\{\frac{\gamma+1}{4}\int\frac{\Delta\sigma}{A}\,dt\right\},$$

since $\Delta\sigma$ cannot be assumed to be small throughout the receding wave; the integral is to be evaluated along an advancing Mach line, and it remains bounded, by (37), even if the path of integration crosses a receding limit line. But $\Delta\lambda = 0$ on the wave front receding into steady flow, and so the solution reduces to

$$L'y = \text{constant on advancing Mach lines}$$

by analogy with (35a), not only on certain narrow strips, but throughout the wave. In contrast to the primary receding wave, the secondary advancing wave is therefore altogether an 'acoustic' wave, and in the *first approximation*, where terms $O(\epsilon)$ are neglected, $\Delta\lambda \equiv 0$.

IV. The second approximation

By (9), (31), and (32),

$$D_+(\Delta s)/Dt = \Delta\sigma, \qquad D_+(\Delta r)/Dt = -v+\rho_+V = O(\epsilon),$$

whence

$$\left.\begin{aligned}\Delta s &= \int\limits_0^\tau \Delta\sigma\,d\tau\\ \Delta r &= O\left(\int\limits_0^\tau \epsilon\,d\tau\right)\end{aligned}\right\}, \tag{43}$$

where the integrals are taken along an advancing Mach line, with τ measured from the wave front travelling into steady flow. A knowledge of $\Delta\sigma$ from the *first approximation* therefore permits us to calculate, as a *second approximation* to the flow, the first-order correction to the velocity and to the speed of sound.

It is convenient to choose as characteristic parameter distinguishing receding Mach lines the time, t_0, at the exit measured from the wave front. By (39), equation (43) may be written

$$\Delta s = \int\limits_0^{t_0} \frac{(M-1)K}{(M_0-1)K_0 z_0}\frac{d\tau_0}{dt_0'}\,dt_0'. \tag{44}$$

From Fig. 2,
$$\left.\begin{aligned} dt &= d\tau+d\tau', \\ dx &= (u+a)\,d\tau+(u-a)\,d\tau' \end{aligned}\right\}, \tag{45}$$
and if both Q and Q' are situated at the exit,
$$(\partial\tau/\partial t)_{x_2=x_0} = d\tau_0/dt_0 = (A_0-U_0)/(2A_0), \tag{46}$$
to the first order, whence the velocity correction Δs becomes
$$\Delta s(x,t) = \frac{-1}{2K_0}\int\limits_0^{t_0} (M-1)K\Delta\sigma_0\,dt_0'; \tag{47}$$
here $t_0 = t-\int\limits_{x_0}^{x} dx/(U-A)$ is the time at which the receding Mach line through (x,t) passes the exit, and $(M-1)K$ is to be taken at the point of intersection of the advancing Mach line through (x,t) with the receding Mach line passing the exit at time t_0', i.e. at the position x' given by
$$\int\limits_{x'}^{x} \frac{A\,dx}{A^2-U^2} = \tfrac{1}{2}(t_0-t_0').$$

Since the *first approximation* does not distinguish between the Mach lines of the unsteady and of the steady flow, it is obvious that the interval is $O(t_0)$ over which x varies on the advancing Mach line between (x,t) and the wave front. If t_0 is small and of the same order as ϵ, at most, $(M-1)K$ may therefore be taken at (x,t), without loss of accuracy, and (47) becomes
$$\Delta s(x,t) = -\frac{[M(x)-1]K(x)}{2K_0}\int\limits_0^{t_0} \Delta\sigma_0\,dt_0'. \tag{47a}$$

For an advancing wave travelling into subsonic, steady flow,
$$\Delta s = O(\epsilon\tau'),$$
$$\Delta r(x,t) = \frac{1}{2L_0}\int\limits_0^{\vartheta_0} (M+1)L\Delta\lambda_0\,d\vartheta_0', \tag{48}$$
where
$$\vartheta_0 = t-\int\limits_{x_0}^{x}\frac{dx}{U+A}$$
is the time at which the advancing Mach line through (x,t) passes the entry of the duct, and $(M-1)L$ is to be taken at the position x'' given by
$$\int\limits_{x''}^{x} \frac{A\,dx}{A^2-U^2} = \tfrac{1}{2}(\vartheta_0'-\vartheta_0).$$
(A suffix 0 is here used to denote values taken at the entry of the duct.)

IV. 1. The equation (33) can be integrated formally even if the assumption that $\Delta\lambda$ may be neglected is abandoned. The integral is still given by (35), except that K has to be replaced throughout by

$$K' = K\exp\left\{\frac{\gamma+1}{4}\int^{t}\frac{\Delta\lambda}{A}\,dt\right\},$$

where the integral is taken along the receding Mach line through the point (x, t). On the other hand, if $\Delta\lambda$ is not neglected,

$$D_{-}(u+a)/Dt - D_{-}(U+A)/Dt = \tfrac{1}{2}(\gamma+1)\Delta\lambda + O(\epsilon),$$

and it follows that (39), and hence also (47), remain valid as they stand. Similarly, if the assumption $\Delta\sigma = O(\epsilon)$ is abandoned in (41), L has to be replaced by

$$L' = L\exp\left\{\frac{\gamma+1}{4}\int^{\vartheta}\frac{\Delta\sigma}{A}\,d\vartheta\right\}$$

throughout (42), but (48) remains valid as it stands.

Equations (47) and (48) can therefore be employed to obtain the *second approximation* even for a region of interaction of two primary waves, where $\Delta\sigma$ and $\Delta\lambda$ must be expected to be of the same order of magnitude. For instance, if a disturbance is set up at the entry of a duct, in a steady flow that is supersonic near the entry, $\Delta\sigma_0$ and $\Delta\lambda_0$ may be assumed to be prescribed as functions of the time at the entry (cf. section II; the suffix 0 is again used to denote values taken at the entry of the duct). Then

$$\left.\begin{aligned} \Delta r(x,t) &= \frac{1}{2L_0}\int_{t_0}^{0}(M+1)L\Delta\lambda_0\,d\vartheta_0' + \Delta r(x_0, t_0) \\ \Delta s(x,t) &= -\frac{1}{2K_0}\int_{0}^{t_0}(M-1)K\Delta\sigma_0\,dt_0' + \Delta s(x_0, \vartheta_0) \end{aligned}\right\}, \tag{49}$$

at any point (x, t) in the region of interaction (Fig. 1), with t_0, ϑ_0, x', and x'' calculated as for (47) and (48). (Note also that τ, in (43), must still be measured from the wave front receding into steady flow, and hence, for the calculation of Δs in the receding wave, after its emergence from the region of interaction, the lower limit of the integrals in (47) and (47 a) must be replaced by the time, T_0, during which the disturbance lasts.)

It should be stressed, however, that the formulae (49) represent only a partial solution of the problem, because the solution given here for the *first approximation* in the region of interaction is only formal. It does not give the limit lines that might occur in the interaction region, nor the distributions of $\Delta\sigma$ and $\Delta\lambda$, respectively, on the Mach lines bordering the interaction region, which would be required as boundary conditions for the

determination of the limit lines that may occur in the primary waves after their emergence from the interaction.

IV. 2. It is now easy to see for which types of waves the theory yields a consistent approximation. Consider first a receding wave generated at the exit of the duct. Equation (47) may be written

$$\Delta s = -\tfrac{1}{2}(M-1)KF(t_0)+\tfrac{1}{2}\int F(t_0')\,d[(M-1)K],$$

where
$$F(t_0) = \frac{1}{K_0}\int\limits_0^{t_0} \Delta\sigma_0\,dt_0'$$

and the integral in the second term is taken along the advancing Mach line, from the wave front to the point where Δs is to be calculated. For a duct of finite length (without sonic throat) the condition

$$|F(t_0)| \ll A_0 \quad \text{for} \quad 0 \leqslant t_0 \leqslant T_0$$

will therefore ensure the $\Delta s_0/S_0 \ll 1$ for $t_0 \leqslant T_0$. Note that the boundary conditions assumed in this investigation do not determine the primary wave beyond the advancing Mach line meeting the wave front at the entry of the duct. The time at which this Mach line passes the exit therefore represents an upper limit for T_0. If ϵ and ϵ' denote respectively the largest values taken by $|F(t_0)|$ and $\left|\int\limits_0^{t_0} F(t_0')\,dt_0'\right|$, the errors in the formulae of the *first approximation* will be $O(\epsilon T)$, at most, where T is the maximum time a receding Mach line spends in the duct; the errors in the formulae of the *second approximation* will be $O(\epsilon' T)$, and $\Delta\lambda$ will be of the same order, at most.

For a primary wave advancing into steady subsonic flow, the corresponding condition is, by (48),

$$|G(\vartheta_0)| \ll A_0 \quad \text{for} \quad 0 \leqslant \vartheta_0 \leqslant T_0,$$

where
$$G(\vartheta_0) = \frac{1}{L_0}\int\limits_0^{\vartheta_0} \Delta\lambda_0\,d\vartheta_0'.$$

No limit on T_0 is implied if the flow is subsonic throughout the duct. For primary waves travelling into steady, supersonic flow, a consistent approximation will be ensured by the same two sets of conditions, by (47), (48), and (49).

On the other hand, the theory will be of interest only if $|\Delta\sigma|$ (or $|\Delta\lambda|$) itself does not remain small at the exit or entry so that shock waves may occur inside the duct, which could not be predicted by Acoustic theory.

Therefore, unless application be restricted to the neighbourhood of wave fronts and to waves generated by disturbances of short duration and extent, the requirement is that the disturbance should be a rapidly fluctuating one. The mean squares of the differences of velocity and pressure from their respective steady-flow values at the exit should be small for the approximation to apply. But the respective mean squares of the fluid acceleration and the local rate of change of pressure, at the exit, should not be small for the theory to be of interest.

This suggests that the theory may be applied to the study of the pressure fluctuations generated by turbulence in the high-speed flow in ducts. These pressure fluctuations are of very small amplitude, and the shock waves to which they give rise are of very small strength, but their study may be of interest if it is desired to distinguish between the velocity fluctuations directly due to local vortices and those due to pressure fluctuations propagated from other parts of the flow field.

It appears from the theory described above that the smallest eddies will generate shock waves after the shortest time. These effects of turbulence, however, will not be followed up here, since it appears to the writer, on the grounds of a rough estimate, that in wind tunnels of the types commonly used today even the shock waves due to the smallest eddies will form only downstream of the working section. More marked effects should, however, be expected from the receding waves in ducts with an entirely subsonic, or an entirely supersonic, mean flow in which a near-sonic velocity is reached in a throat.

V. The shock path

Knowledge of $\Delta\sigma$ from the *first approximation* permits us also to calculate the shock-path in the (x, t)-plane, to the first order in the shock strength. The example chosen is again that of a receding wave, generated at the exit of a duct.

At any point of the shock-path two receding Mach lines meet. Let a suffix s refer to the shock-point, and the suffixes 1, 2 refer respectively to the upstream and downstream receding Mach lines through the shock-point; in particular, let t_1 and t_2 denote the respective times at which these Mach lines pass the exit.

It is a well-known consequence of the Rankine–Hugoniot shock equations that, to the first order in the shock strength, the speed of propagation of a shock is the arithmetic mean of the speeds of propagation of signals upstream and downstream of the shock. More precisely, the path of a receding shock bisects, at every point, the angle between the receding Mach lines at the point (and similarly for an advancing shock and the advancing

Mach lines). Hence, to the first order, $AB = BC$ in Fig. 3, and so, if t_0 denotes the time at which a receding Mach line passes the exit, then at the shock (Fig. 3),

$$\{(\partial x/\partial t_0)_{t_s}\, dt_0/dt_s\}_2 + \{(\partial x/\partial t_0)_{t_s}\, dt_0/dt_s\}_1 = 0, \tag{50}$$

where the suffix t_s denotes that the derivative is taken at constant time, $t = t_s$.

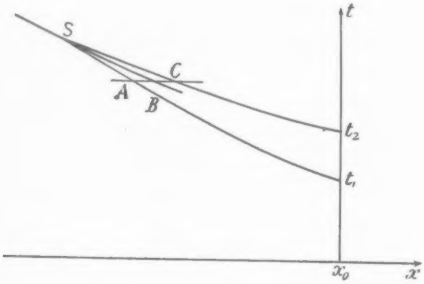

FIG. 3.

Another well-known property of a shock is that, even to the second order in the strength, there is no change of entropy across it. This means that the shock-path fits into the solution of the equations of isentropic flow in the following sense. The isentropic solution possesses a limit line with a cusp at the point where the shock-path starts.† Between the two branches of the limit line, the mapping of the characteristic plane on to the (x, t)-plane is three-sheeted (**5, 6**). The shock-path lies in this region of the (x, t)-plane, and the flow passes across the shock from the first to the third sheet. Suppose the shock-path to be marked on both the first and the third sheet. Then, as a line t = constant $= t_s$ is followed on the first sheet from the shock to the limit line, then on the second sheet back to the other branch of the limit line, and finally on the third sheet from the limit line to the shock, the total change in x must be zero. Points on this line $t = t_s$ may be regarded as different if they lie on different sheets, even if they seem to coincide in the (x, t)-plane, and there is then a $(1, 1)$ correspondence between the points and the receding Mach lines crossing this line, and thereby also between these points and the respective times at which receding Mach lines pass the exit. The total change in x may therefore be written

$$\int_{t_1}^{t_2} (\partial x/\partial t_0)_{t_s}\, dt_0 = 0. \tag{51}$$

† The possibility of the occurrence of an edge of regression is ignored for the moment, and that case is discussed in section V. 1 below.

By (39), (45), (46), and (35),

$$(\partial x/\partial t_0)_{t_s} = -a(M-1)Kz/(K_0 z_0) = -a(M-1)[1+I/(K_0 z_0)], \tag{52}$$

where all the quantities on the right-hand side are taken on the same receding Mach line; those with a suffix 0 at the exit, and those without a suffix at time t_s. If x_a and x_b denote respectively the position of the limit points on the line $t = t_s$, and θ the greatest value of $|\partial x/\partial t_0|$ on this line, between the limit points, then $|x_a - x_b| = O\{\theta(t_2-t_1)\}$, which is small compared with θ. To the first order, therefore, quantities denoted by capitals without a suffix, in (52), may be regarded as constants in the integrand of (51). Moreover, to the first order, Δa may be neglected in comparison with A. It then follows from (51) and (52) that

$$\frac{1}{I(x_s)} = \frac{-1}{(t_2-t_1)K_0}\int_{t_1}^{t_2} \Delta\sigma_0\, dt_0, \tag{53}$$

i.e. the position of the shock-point where the receding Mach lines that pass the exit at times t_1 and t_2 respectively meet, is given directly by the mean value of $\Delta\sigma$ at the exit between t_1 and t_2.

Elimination of I from (52), by the help of (53), substitution in (50), and integration gives

$$\int_{t_1}^{t_2} dt \int_{C}^{t} \Delta\sigma_0\, dt_0 - \tfrac{1}{2}(t_2-t_1)\left(\int_{C}^{t_2} + \int_{C}^{t_1}\right)\Delta\sigma_0\, dt_0 = 0 \tag{54}$$

where C is arbitrary. From that relation, one of t_2, t_1 can be determined when the other has been chosen.

The strength of the shock is given by

$$\delta \equiv (p_2-p_1)/P = \gamma(s_2-s_1)/A + O(\delta^2)$$

(where all quantities are taken at $x = x_s$, $t = t_s$), as can be shown without difficulty from the shock-equations (7). Since S depends only on x, we may evaluate the jump of Δs across the shock, instead of the jump of s. The respective shock-points on the first and third sheet do not lie on the same advancing Mach line, but the rate of change of Δs in the receding Mach direction is $O(\epsilon)$ by (10) and (31), and hence, to the first order at least,

$$\delta = (\gamma/A)\int_{t_1}^{t_2} \Delta\sigma(d\tau/dt_0)\, dt_0,$$

where all the quantities in the integrand are taken on the same receding Mach line; those with a suffix 0 are taken at the exit, those without a suffix on the advancing Mach line through either of the two shock-points. The quantities depending only on the value of x on this advancing Mach line

may again be regarded as constants during the integration, whence, by (39) and (45),

$$\delta = -\frac{\gamma(M-1)K}{2AK_0}\int\limits_{t_1}^{t_2} \Delta\sigma_0\,dt_0. \tag{55}$$

The absolute speed of propagation of the shock is

$$dx_s/dt_s = \tfrac{1}{2}(u_s-a_s)_1+\tfrac{1}{2}(u_s-a_s)_2,$$

by the property of the arithmetic mean referred to at the beginning of this section. By (43), (44), and (47 a), this may be written

$$\frac{dx_s}{dt_s} = U-A+\frac{\gamma+1}{8}(M-1)\frac{K}{K_0}\left(\int\limits_0^{t_2}+\int\limits_0^{t_1}\right)\Delta\sigma_0\,dt_0, \tag{56}$$

where the use of (47 a) implies the assumption that t_2 and t_1 are small.

For an advancing wave, the formulae corresponding to (53) to (56), respectively, are

$$\frac{1}{J(x)} = \frac{1}{t_1-t_2}\frac{1}{L_0}\int\limits_{t_2}^{t_1}\Delta\lambda_0\,dt_0,$$

$$\int\limits_{t_2}^{t_1} dt \int\limits_C \Delta\lambda_0\,dt_0-\tfrac{1}{2}(t_1-t_2)\left(\int\limits_C^{t_2}+\int\limits_C^{t_1}\right)\Delta\lambda_0\,dt_0 = 0,$$

$$\delta = -\frac{\gamma(M+1)L}{2AL_0}\int\limits_{t_2}^{t_1}\Delta\lambda_0\,dt_0,$$

and

$$\frac{dx_s}{dt_s} = U+A+\frac{\gamma+1}{8}(M+1)\frac{L}{L_0}\left(\int\limits_0^{t_2}+\int\limits_0^{t_1}\right)\Delta\lambda_0\,dt_0,$$

where t_1, t_2 are the times at which the advancing Mach lines meeting the shock at $x = x_s$, $t = t_s$ on the upstream and downstream side respectively, pass the line $x = x_0$ on which $\Delta\lambda_0$ is prescribed (cf. section II).

V. 1. It should be noted that equations (53) and (54) are formally identical with the equations first obtained by Whitham (2) for the shape of the shock in his first-order theory of the steady, supersonic, axially symmetrical flow past a slender body of revolution. The simple geometrical interpretation given by Whitham may therefore also be applied to equations (53) and (54). This suggests that similar results may be obtainable also for a number of other problems. The arguments used above to establish (50) and (51) apply to any weak shock separating two regions where the equations of the flow are hyperbolic. To the first order of the shock strength,

the shock equations may therefore be written in the form (50), (51) in all such cases. Their integration in the form (53), (54) would then appear to depend essentially on whether the integrand of (51) may be written in the form

$$\phi_1(x_s, t_s)\psi_1(t_0)+\phi_2(x_s, t_s)\psi_2(t_0),$$

where t_0 is a characteristic parameter. For, to the first order, the variation of the ϕ_i $(i = 1, 2)$ between the two branches of the limit line can always be neglected in the integrand of (51). If we write, for brevity,

$$[f]_1^2 \quad \text{for} \quad f(t_2)-f(t_1), \qquad \{f\}_1^2 \quad \text{for} \quad f(t_2)+f(t_1),$$

and

$$\int^t \psi_i \, dt_0 = F_i(t),$$

it follows from (51) that

$$\phi_2[F_2]_1^2+\phi_1[F_1]_1^2 = 0,$$

which has the same form as (53); and using this to eliminate the ϕ_i from (50), we obtain

$$[F_1]_1^2\{dF_2\}_1^2-[F_2]_1^2\{dF_1\}_1^2 = d([F_1]_1^2\{F_2\}_1^2)-2[F_2\,dF_1]_1^2 = 0$$

which gives (54).

It should also be noted, however, that for the problem studied in this paper the advantage of writing the equations for the shock in the general form (53) to (56) lies in their covering a number of different cases rather than in the degree of accuracy to which these equations and their geometrical interpretation hold in any one case. This is best explained at the instance of two examples.

Suppose first that $\Delta\sigma_0$ is a continuous function of t_0, and analytic for $t_0 > 0$. As explained in section III. 4, the point where the shock-path starts lies on the receding Mach line passing the exit at the time, t_0^* say, when $\Delta\sigma_0$ reaches its highest value. Therefore, for small values of $t_0' = t_0-t_0^*$,

$$\Delta\sigma_0 = \Delta\sigma^*+\tfrac{1}{2}\beta t_0'^2+\tfrac{1}{6}\eta t_0'^3+\dots \quad \text{and} \quad \beta < 0. \tag{57}$$

When substituting this into (53) to (56), we must remember that all but the largest term was neglected in the integrand of (51). It follows that (54) holds only in so far as the largest term on the left-hand side is concerned, and with $\Delta\sigma_0$ given by (57), this implies that

$$(\beta/24)(t_1'+t_2')(t_2'-t_1')^3 = 0,$$

from which we conclude (cf. Fig. 3) that

$$t_1' = -t_2'+O(t_2'^2). \tag{58}$$

This shows that our approximation does not permit us to take account of terms $O(t'_0)^3$ in $\Delta\sigma_0$; but if terms $O(t_0'^3)$ are neglected, it is immediately evident from Fig. 3, for reasons of symmetry, that (58) follows from the assumption of (57). By (57), (58), (53), (40), and (55),

$$I(x_s)-I(x^*) = \beta I(x^*)t_2'^2/(6K_0)+O(t_2'^3), \tag{59}$$

$$\delta = \gamma(1-M)K\Delta\sigma^* t'_2/(AK_0)+O(t_2'^2), \tag{60}$$

where x^* is the position at which the shock-path starts. By (56), (57), (58), (35), and (40), the time at which the shock reaches position x_s is, to the first order,

$$t_s(x_s) = t_s(x^*)+\int\limits_{x_s}^{x^*}\frac{dx}{A-U}+[1-I(x_s)/I(x^*)]\int\limits_0^{t_0^*}(\Delta\sigma_0/\Delta\sigma^*)\,dt_0. \tag{61}$$

Secondly, suppose that $\Delta\sigma_0$ is discontinuous at $t_0 = 0$ and, say,

$$\left.\begin{aligned}&\Delta\sigma_0 \equiv 0 \quad \text{for} \quad t_0 < 0,\\ &\Delta\sigma_0 = \Delta\sigma^*+\alpha t_0+\tfrac{1}{2}\beta t_0^2+\dots \quad \text{and} \quad \alpha < 0, \quad \text{for} \quad t_0 > 0\end{aligned}\right\}. \tag{62}$$

Then the shock starts on the wave front, and one of the two branches of the limit line is replaced by an edge of regression (**6**), at which the first and second sheets of the mapping meet. The arguments employed in the preceding section to derive equations (50) to (56) remain valid, however, if the words 'limit line' and 'limit point' are replaced respectively by 'edge of regression' and 'point of regression', wherever appropriate, and this case is therefore also covered by (53) to (56). On substituting (62), remembering that only the largest term in each expression is reliable, we find

$$\tfrac{1}{2}\Delta\sigma^* t_1 t_2 = 0,$$

from (54), since now $t_1 \leqslant 0$ (Fig. 3), and

$$K_0/I(x^*)-K_0/I(x_s) = \tfrac{1}{2}\alpha(t_1+t_2),$$

from (53). Since the shock travels in the direction of x decreasing, $x_s \leqslant x^*$, and hence, by (35) and since $\alpha < 0$, $(t_1+t_2) > 0$; we conclude that

$$t_1 = O(t_2^2) \tag{63}$$

at most, and
$$I(x_s)-I(x^*) = \alpha I(x^*)t_2/(2K_0)+O(t_2^2); \tag{64}$$

also, from (55), (56), and (62) to (64), we have

$$\delta = \gamma(1-M)K\Delta\sigma^* t_2/(2AK_0)+O(t_2^2), \tag{65}$$

$$t_s(x_s) = t_s(x^*)+\int\limits_{x_s}^{x^*}\frac{dx}{A-U}+\Delta\sigma^*[1-I(x_s)/I(x^*)]^2/(4\alpha^2). \tag{66}$$

Equations (63) to (66) should be compared with (58) to (61).

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SUPERSONIC THEORY OF DOWNWASH FIELDS

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SUMMARY

Two methods have so far been proposed for the calculation of downwash fields behind a supersonic wing on the linearized theory. That due to Lagerstrom (**1**) is an extended conical flow method, while Mirels and Haefeli (**2**) and Robinson and Hunter-Tod (**3**) replace the wing by an equivalent vortex distribution. Since references (**1**) and (**2**) are not readily available in this country, their contents are briefly summarized.

We then show that the basic formula of Robinson and Hunter-Tod (**3**), giving the downwash in terms of the flow on the wing, can be deduced very rapidly from Hadamard's theorem. When the flow is conical, this equation can be shown to be identical with that of Lagerstrom. Also, this formula makes clear the nature of the approximation made by Mirels and Haefeli. Finally, an asymptotic expansion of the integral kernel occurring in the equation gives a 'practical' method of calculating the downwash behind any wing.

1. Introduction: Lagerstrom's method

THE determination of the flow field behind a wing is of considerable practical importance, and also of some mathematical interest. If we take a rectangular coordinate system such that x is the direction of the free stream, while the wing lies in the plane $z = 0$, then the components v $(= q_y)$ and w $(= q_z)$ of the velocity q are called the sidewash and downwash respectively.

Calculations of supersonic downwash have so far been made using the linearized theory. This is unfortunate, since two important effects, the rolling-up and downward deflexion of the trailing vortex wake, are then missed. However, even using the linearized theory, the formulae obtained are rather involved, and work with more accurate equations will lead to even greater complications. Also, since the linearized theory gives good results for thin wings at small angles of incidence, it would seem that viscous effects should be considered before those of non-linearity.

Throughout this paper we use the rectangular wing-tip for purposes of illustration, but the arguments are obviously general.

Within the Mach cone OAB, the velocities are known functions of the complex conical coordinate b, with O as origin. This coordinate is defined by

$$\frac{\beta y}{x} = r\cos\theta, \qquad \frac{\beta z}{x} = r\sin\theta, \qquad b = \frac{\cos\theta - i(1-r^2)^{\frac{1}{2}}\sin\theta}{r},$$

where $\beta = \sqrt{(M^2-1)}$. Throughout this paper the quantities which we describe as velocities are, in fact, the ratios of the physical perturbation velocities to the free-stream velocity. Also, since the problem is anti-symmetric in z, we shall only consider the region $z \geqslant 0$. In particular, we shall have

$$u = \frac{\partial\phi}{\partial x} = L(b), \tag{1}$$

ϕ being the perturbation velocity potential (we use the symbol L since this

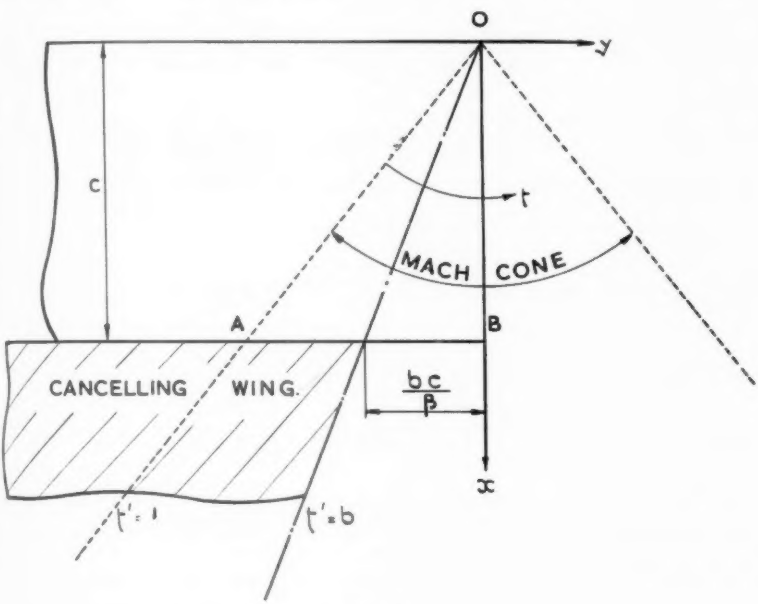

FIG. 1.

quantity determines the lift of the wing). In the case of rectangular wing-tip at incidence α,

$$L(b) = \frac{2\alpha}{\pi\beta}\mathrm{re}(\sin^{-1}\sqrt{b}).$$

This formula cannot be valid downstream of the Mach wave from the trailing edge, since it gives $u \neq 0$ on the trailing wake. Lagerstrom pointed out that, by the superposition of certain elementary conical flows, this excess lift could be cancelled, and that the actual downwash field would then be the sum of the flows due to the physical wing and the 'cancelling wings'.

The conical flows treated in normal wing theory give constant downwash

over the wing, but Lagerstrom's cancelling wing is one of constant lift. In the notation of Goldstein and Ward (**4**) it is specified by

$$\begin{aligned} \mathrm{re}(G_3') &= \bar{u}_3 \quad (t < b), \\ &= 0 \quad (t > b). \end{aligned}$$

Using the methods of reference (**4**), it can be shown that the downwash due to this wing is given by

$$\frac{u_2}{\beta \bar{u}_3} = \mathrm{re}\{G(b,t)\},$$

where $$G(b,t) = \frac{1}{\pi}\left\{\frac{1}{b}\ln|T| + 2\tan^{-1}T + \frac{B^2-1}{2B}\ln\left|\frac{T-B}{1-BT}\right| - \tfrac{1}{2}\pi\right\}, \tag{2}$$

$$T = \frac{1-\sqrt{(1-t^2)}}{t}, \qquad B = \frac{1-\sqrt{(1-b^2)}}{b}.$$

In the plane of the wing, t becomes a real coordinate, and

$$\frac{u_2}{\beta \bar{u}_3} = G(b,t).$$

Lagerstrom then finds that the downwash field of the wing-tip in the plane $z = 0$ may be written

$$w = \beta \int_0^1 \{G(b,t) - G(b,t^*)\}\frac{\partial L}{\partial b}\,db, \tag{3}$$

where $$t^* = \frac{\beta y/c - b}{x/c - 1}. \tag{3a}$$

(A similar formula may be written down for a general station.) Lagerstrom also shows that the flow in the Trefftz plane ($x \to \infty$) is particularly simple; he has published an account of this work (**5**).

2. Vortex-distribution methods

Any lifting surface can be replaced by a system of bound vortices. Since it also trails a vortex wake behind it, the calculation of the resultant downwash is equivalent to finding the flow due to a given vortex system. Using this equivalence, Robinson and Hunter-Tod (**3**) deduce the formula which we derive in section 3 from Hadamard's theorem. These formulae are just as complicated as Lagerstrom's.

Mirels and Haefeli (**2**) therefore replace the lifting surface of the wing by a lifting line, and calculate the downwash due to this. Their formulae are therefore approximate, but the error is not important for stations more than a chord downstream of the trailing edge. The equations are much more tractable than those of Lagerstrom, so that their paper represents a very definite contribution to the problem of finding downwash fields.

The strength of Mirels and Haefeli's line is variable, being equal to $\Gamma(y)$, the local circulation round the wing. Since the perturbation velocity potential ϕ is antisymmetric in z, and is zero on l, the leading edge of the wing, Γ is twice ϕ_t, t denoting the trailing edge. Logically the lifting line should be the locus of local centres of pressure, but this gives a non-linear curve, which is nearly as difficult to handle as a lifting surface. The line is

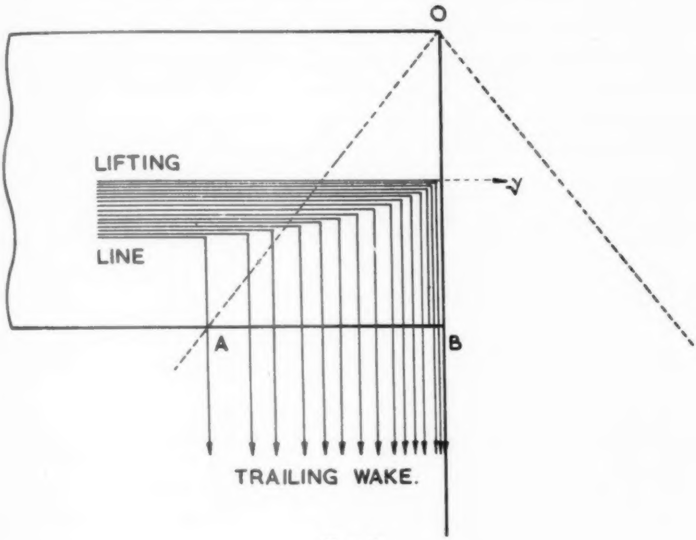

FIG. 2.

therefore taken to be straight, and generally passes through the centre of pressure of the wing. For representing a triangular or swept-back wing, a lifting line with a single bend is advantageous, but for a rectangular wing-tip Mirels and Haefeli take the line to be straight, and normal to the direction of the free stream.

Now vorticity must be conserved, so that the lifting line will trail a vortex wake behind it in the manner indicated in Fig. 2; the vorticity shed from the main line between y and $y+dy$ is equivalent to a line vortex of constant strength $(\partial\Gamma/\partial y)\,dy$. Using the methods of Robinson (**6**), Mirels and Haefeli show that the downwash field due to this vortex system is

$$w(\mathbf{r}) = -\frac{1}{2\pi}\int\limits_{y_1}^{y_2} K(0)\frac{\partial\Gamma}{\partial y'}dy', \tag{4}$$

$$K(x) = \frac{(x-x')(y-y')(R^2-\beta^2z^2)}{R\{(x-x')^2-\beta^2z^2\}\{(y-y')^2+z^2\}}, \tag{5}$$

where $$R^2 = (x-x')^2-\beta^2\{(y-y')^2+z^2\},$$

y_1 and y_2 are the coordinates of A and B, and we have taken the y-axis to coincide with the lifting line; $\partial\Gamma/\partial y$ is easily calculated for most wings of practical importance: for the rectangular wing-tip at incidence α

$$\frac{\partial\Gamma}{\partial y} = -\frac{4\alpha}{\pi}\sqrt{\left(\frac{c-\beta y}{\beta y}\right)}.$$

The computations of Mirels and Haefeli were made with formula (4), and were in consequence rather complicated. However, we should not expect the formula to hold for small x, since in obtaining it we have distorted the lift distribution of the wing (this will be made clear in the next section). This suggests that we should consider an asymptotic expansion of (4) in powers of x. Now

$$K(0) \sim \frac{y-y'}{(y-y')^2+z^2}-\frac{\beta^2(y-y')}{2x^2}-\frac{\beta^4(y-y')\{(y-y')^2+9z^2\}}{8x^4}+O(x^{-6}) \quad (6)$$

so that the required expansion of (4) is

$$w = -\frac{1}{2\pi}\int\limits_{y_1}^{y_2}\frac{y-y'}{(y-y')^2+z^2}\frac{\partial\Gamma}{\partial y'}\,dy'-\frac{\beta^2}{4\pi x^2}\int\limits_{y_1}^{y_2}(y-y')\frac{\partial\Gamma}{\partial y'}\,dy'+O(x^{-4}). \quad (7)$$

We shall see later that the subsequent terms of this expansion are not reliable. The first term represents the flow in the Trefftz plane, and was discovered by Mirels and Haefeli using a different method; we shall now show that this term does indeed give the same flow as Lagerstrom's method. Repeating the above argument, we can show that the sidewash in the Trefftz plane is given by

$$v = \frac{1}{2\pi}\int\limits_{y_1}^{y_2}\frac{z}{(y-y')^2+z^2}\frac{\partial\Gamma}{\partial y'}\,dy'. \quad (8)$$

In both (7) and (8) the integral can be taken from $-\infty$ to $+\infty$ if we take $\partial\Gamma/\partial y = 0$ if $y < y_1$ or $> y_2$ (this is consistent with the definition of Γ). Combining these two equations, we can write

$$v-iw = \frac{1}{2\pi i}\int\limits_C\frac{F(\zeta')\,d\zeta'}{\zeta-\zeta'},$$

where the complex variables ζ, ζ' are defined by

$$\zeta = y+iz, \qquad \zeta' = y'+iz'.$$

The contour C consists of the real axis from $-R$ to $+R$, together with the semicircle $Re^{i\theta}$ $(0 \leqslant \theta \leqslant \pi)$ as $R \to \infty$; the real axis must be cut between

y_1 and y_2 before the integral can be defined: the path goes along the upper lip of the cut. Thus,

$$v-iw = F(\zeta). \tag{9}$$

Now $F(\zeta')$ is an analytic function whose real part is equal to $\partial\Gamma/\partial y'$ on $y' = 0$ and which behaves suitably at infinity (this is necessary to make the integral round the semicircle converge). (9) is therefore a statement of Lagerstrom's procedure (**5**).

The second term is simple to calculate, but the important point is that it is independent of z; hence if we know the flow in the Trefftz plane and in the plane of the wing, it can at once be found wherever this expansion is valid. This holds for stations not too near the Mach wave of the trailing edge.

3. Application of Hadamard's theorem

Hadamard (**7**) has shown that if the sources of a disturbance are confined to the plane $z = 0$, then the appropriate solution of the linearized equation

$$\beta^2\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

is

$$\phi(\mathbf{r}) = -\frac{\beta^2}{\pi}{}^{*}\!\!\iint \phi(x',y',0)\frac{z}{R^3}\,dx'dy', \tag{10}$$

R being defined by equation (5). The symbol ${}^{*}\!\!\iint$ indicates that the finite part of the integral is to be taken; the region of integration is that part of the plane $z = 0$ lying within the forecone of the point r. Integrating (10) by parts, Ward (**8**, **9**) has obtained the formula

$$\phi(\mathbf{r}) = \frac{1}{\pi}\iint\frac{\partial\phi}{\partial x'}\frac{z(x-x')}{\{(y-y')^2+z^2\}R}\,dx'dy', \tag{11}$$

$\partial\phi/\partial x$ is zero except on the wing, so that the region of integration is that part of the wing lying within the forecone of $\mathbf{r}$. He suggests (11) as being suitable for downwash calculations, but we do not regard the formula as being well adapted to this purpose. It does not give w directly, and when differentiated with respect to z, it gives rise to another divergent integral, which must be treated by the finite part technique.

If we integrate (11) by parts again, we find

$$\phi(\mathbf{r}) = -\frac{1}{\pi}\iint\frac{\partial^2\phi}{\partial x'\partial y'}\tan^{-1}\left\{\frac{zR}{(x-x')(y-y')}\right\}dx'dy'$$

and hence

$$w = \frac{\partial\phi}{\partial z} = -\frac{1}{\pi}\iint\frac{\partial^2\phi}{\partial x'\partial y'}K(x')\,dx'dy', \tag{12}$$

where $K(x)$ is defined by formula (5). Again, the integral is over that part of the wing which lies within the forecone of $\mathbf{r}$. This integral is finite, which

is a practical convenience. This formula is equivalent to Robinson and Hunter-Tod's formulae (30), (33).

We shall now show that Lagerstrom's formula (3) is a particular case of (12) in which the flow on the wing is conical. If we confine ourselves to the plane of the wing, (12) becomes

$$w = -\frac{1}{\pi}\iint \frac{\partial^2\phi}{\partial x'\partial y'}\,\frac{\sqrt{\{(x-x')^2-\beta^2(y-y')^2\}}}{(x-x')(y-y')}\,dx'dy'. \tag{12a}$$

Since the flow on the wing is conical,

$$\frac{\partial\phi}{\partial x'} = L(b), \qquad \frac{\partial^2\phi}{\partial x'\partial y'} = \frac{\beta}{x'}\frac{\partial L}{\partial b},$$

where $b = \beta y'/x'$ (identical with Lagerstrom's variable). If we also use Lagerstrom's other conical coordinate $t = \beta y/x$ (12a) becomes, for a rectangular wing-tip

$$w = \beta\int\limits_0^1 I(b,t)\frac{\partial L}{\partial b}\,db, \tag{13}$$

where
$$I(b,t) = -\frac{1}{\pi}\int\limits_0^{c/x}\frac{\sqrt{\{(1-\xi)^2-(t-b\xi)^2\}}}{(1-\xi)(t-b\xi)}\,d\xi$$

($\xi = x'/x$). It is easy to show that

$$I(b,t) = G(b,t)-G(b,t^*),$$

so that (13) is identical with Lagerstrom's formula (3). Also (12) will, in principle, give the downwash field of any wing exactly (subject to the limitations of the linearized theory) while Lagerstrom's method is restricted to wings on which the flow is conical. However, on other wings, the integrals are likely to be rather troublesome.

To handle (12) in general, some approximation must be made, and the most obvious one is to set

$$\phi(x',y') = \phi_t(y')H(x'), \tag{14}$$

where $\phi_t(y')$ denotes the value of ϕ on the trailing edge and $H(x')$ is the unit function

$$H(x') = 0 \quad (x' < 0),$$
$$= 1 \quad (x' > 0).$$

This replaces the smooth increase of ϕ from l to t by a sudden jump across the y-axis. From (14)

$$\frac{\partial\phi}{\partial x'} = \phi_t(y')\,\delta(x') = \tfrac{1}{2}\Gamma(y')\,\delta(x'), \tag{15}$$

where $\Gamma(y')$ is the local circulation round the wing, and $\delta(x')$, the Dirac

delta function, is the derivative of the unit function. Hence this approximation is equivalent to replacing the wing by a lifting line along $x' = 0$, the precise analogue of Mirels and Haefeli's procedure. From (15)

$$\frac{\partial^2\phi}{\partial x'\partial y'} = \frac{1}{2}\frac{\partial\Gamma}{\partial y'}\,\delta(x'),$$

so that (12) becomes
$$w = -\frac{1}{2\pi}\int\limits_{y_1}^{y_2} K(0)\frac{\partial\Gamma}{\partial y'}\,dy',$$

identical with Mirels and Haefeli's formula (4).

It is difficult to make a general estimate of the error of the approximation (13). But if we consider an asymptotic expansion of the exact formula (12), this can be compared with the expansion (7), derived from (4). We find

$$K(x') \sim \frac{y-y'}{(y-y')^2+z^2} - \frac{\beta^2(y-y')}{2x^2} + \frac{\beta^2(y-y')x'}{x^3} - \\ - \frac{\beta^2(y-y')\{4x'^2+\beta^2(y-y')^2+9\beta^2z^2\}}{8x^4} + O(x^{-5}), \tag{16}$$

so that the asymptotic form of (12) is

$$w \sim -\frac{1}{\pi}\int\limits_{\text{span}} \left\{\frac{(y-y')}{(y-y')^2+z^2} - \frac{\beta^2(y-y')}{2x^2}\right\} dy' \int\limits_{l}^{t} \frac{\partial^2\phi}{\partial x'\partial y'}\,dx' + \\ + \frac{\beta^2}{\pi x^3}\int\limits_{\text{span}} (y-y')\,dy' \int\limits_{l}^{t} x'\,\frac{\partial^2\phi}{\partial x'\partial y'}\,dx' + O(x^{-4}). \tag{17}$$

Now
$$\int\limits_{l}^{t} \frac{\partial^2\phi}{\partial x'\partial y'}\,dx' = \left(\frac{\partial\phi}{\partial y'}\right)_t = \frac{1}{2}\frac{\partial\Gamma}{\partial y'}$$

so that the first two terms of (17) are identical with those of (7), which is therefore accurate, in general, to $O(x^{-3})$. If the axes are chosen so that

$$\int\limits_{\text{span}} (y-y') \int\limits_{l}^{t} x'\,\frac{\partial^2\phi}{\partial x'\partial y'}\,dx' = 0, \tag{18}$$

then these first two terms will be accurate to $O(x^{-4})$; the line which satisfies (18) will not usually pass through the centre of pressure. Now in their paper Mirels and Haefeli represent a 'narrow' triangular wing by an unbent lifting line; they get best agreement with the known downwash field when the line is placed at the $\frac{3}{4}$-chord point, while the centre of pressure is at the $\frac{2}{3}$-chord point. The above remark accounts for this phenomenon.

We therefore claim that the expansion (7) will give the downwash field

behind a rectangular wing to the same accuracy as the much more complicated calculations of Mirels and Haefeli, while the expansion (17) should give better accuracy, since the term in x^{-3} is not taken into account in Mirels and Haefeli's work. Calculations to check this claim are being started, and will be reported elsewhere.

In the above calculations, we have tacitly assumed that $\partial\phi/\partial y$ is continuous across the leading and trailing edges, so that they are strictly applicable to rectangular wings only. It is quite easy to take discontinuities into account, and we find that the first two terms of (17) are still accurate if $\partial\Gamma/\partial y$ is taken to be twice the sidewash immediately behind the Mach wave from the trailing edge. Robinson and Hunter-Tod (**3**) have given explicitly the correction necessary for a discontinuity, so that we need not repeat the calculation.

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SOME UNSTEADY MOTIONS OF A SLENDER BODY THROUGH AN INVISCID GAS

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SUMMARY

A source method is used to solve some problems of unsteady motion of a slender body through a perfect, inviscid, compressible fluid. The effects of variations in the forward velocity and in the angle of incidence are investigated on a linearized theory. The results are established for slender bodies both of revolution and of general cross-section. The extra aerodynamic force components produced by linear and angular acceleration are found to be small compared with the corresponding aerodynamic force components in steady motion, provided that the changes in linear and angular velocity, in the time the body takes to travel its own length, are small compared with the actual linear and angular velocities respectively. A similar result is obtained for the effect of angular velocity.

Introduction

A LINEARIZED potential theory is used to solve the problem of the motion of a slender body through a perfect, inviscid gas. First approximations to the aerodynamic forces are obtained for various prescribed, unsteady velocities of the body. In this way the effect of both linear and angular acceleration on the aerodynamic force is investigated for oscillations of small amplitude.

The work, which is developed by a source method, deals with the slender body of revolution in Part I, whilst the slender body of general cross-section is considered in Part II. In each case the effects of variable forward speed and incidence are investigated.

For a slender body, the thickness ratio, ϵ, defined as the ratio of the maximum diameter of the body to its length, and the angle between the tangent to the body surface and the direction of motion must both be small and of the same order. The condition that the curvature at any point on the contour of the body in a transverse plane of cross-section be at most $O(1/\epsilon)$ excludes very small outwardly-convex radii of curvature. The nose of the body is taken to be pointed so that attached shocks only occur; the third-order entropy change across these shocks will not affect this linearized theory.

In a system with unit acoustic velocity it is assumed that the axial velocity of the body is $O(1)$ and that the cross-flow velocity is less than 0·4.

If these conditions are satisfied, then the fluid perturbation velocity will be everywhere small compared with the acoustic velocity, and a linear method of approach will be sufficiently accurate to first order in ϵ, provided that the body accelerates rapidly through the transonic region. The fluid perturbation velocity is the gradient of a scalar potential function, ϕ, which satisfies the wave equation on a linearized theory. The linearized potential, ϕ_0, at points distant r from the axis of the body is obtained as a descending series in r, each term of which represents a source or multi-source potential satisfying Laplace's equation in two dimensions. Thus, the flow is essentially two-dimensional in any plane transverse to the axis of the body.

A set of cylindrical polar coordinates (r, θ, η), referred to axes moving with the body is taken, with η measured along the axis of the body from its base, and r and θ are polar coordinates in the plane η = constant. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a triad of unit orthogonal vectors, with $\mathbf{i}$ along the axis of the body, and $\mathbf{j}$ in the vertical plane through $\theta = 0$. The corresponding fixed, cylindrical polar coordinates (r_1, θ_1, s_1), have their polar axis along the direction of mean motion, with $s_1 = 0$ at the initial position of the base of the body. The displacement of the base of the body along the fixed polar axis after a time t from rest is $s_1 = f(t)$. The initial condition that the body starts from rest at $t = 0$ gives

$$f(0) = f'(0) = 0. \tag{1}$$

The corresponding fixed triad of unit orthogonal vectors, $(\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1)$, has $\mathbf{i}_1$ along the fixed polar axis and $\mathbf{j}_1$ in the plane $\theta_1 = 0$.

If, in addition to the above assumptions, the fluid is also non-conducting, then variables p, ρ, and $\mathbf{u}$, representing the pressure, density, and perturbation velocity respectively, satisfy the equations

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0, \tag{2}$$

and
$$\frac{\partial \phi}{\partial t} + \int \frac{dp}{\rho} + \tfrac{1}{2}u^2 = \text{a spatial constant}, \tag{3}$$

where
$$\mathbf{u} = \nabla \phi. \tag{4}$$

Also for a perfect gas whose motion is isentropic,

$$p = K\rho^{\gamma}, \tag{5}$$

where K and γ are known constants. In a system with unit acoustic velocity, these equations reduce to the linearized equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2}. \tag{6}$$

It is well known that a solution to (6) is the source potential

$$\frac{q(s_1', t')}{R_1}, \quad \text{where } t' = t - R_1, \text{ and } R_1^2 = (s_1 - s_1')^2 + r_1^2. \tag{7}$$

This source, at a point $(0, 0, s_1', t')$ in a four-dimensional space-time continuum, gives rise to a disturbance at the point (r_1, θ_1, s_1, t) where $t = t' + R_1$.

For motion at a variable incidence of small magnitude, the sources are placed on the axis of the body, and, since $s_1 - f(t) = \eta(1 + O(\epsilon^2))$, the source potential is

$$\frac{q(s', t')}{R}\{1 + O(\epsilon^2)\},$$

where $\quad t' = t - R, \qquad R^2 = (s - s')^2 + r^2, \qquad s = \eta + f(t). \qquad (8)$

Clearly $\dfrac{\partial^n}{\partial x^n}\left\{\dfrac{q(s', t')}{R}\right\}$ is also a solution of (6) to the same order; for $n = 1$ this is a doublet potential, whilst in general it represents an nth-order multisource potential. The general solution to (6) is obtained from a combination of source and multisource potentials of all orders. The multisource densities are obtained from the boundary condition stipulating that the normal velocity of the fluid and of the body must be equal on Σ, the surface of the body.

Although a source method has been used in this paper, the series for the potential at points near the axis of the body may be obtained by an operational method (see Ward (**3**)).

Part I

SLENDER BODY OF REVOLUTION

A. Direct motion

1. *Solution for the velocity potential*

This section is concerned with the problem of the motion of a slender body of revolution along its own geometrical axis at a variable forward speed, $f'(t)$. The motion is represented by a line of sources placed along the axis of the body; these sources give the correct fluid displacements provided that the boundary condition on Σ is satisfied. The total potential is a sum of elementary source potentials, viz.

$$\phi_1(r, \theta, s, t) = \int_G \frac{q_1(s', t')}{R}\, ds', \tag{9}$$

where the region of integration, G, includes all points on the axis of the body at which there is a source at a time $t' = t - R$ (an implicit equation).

Since the aerodynamic force is obtained from the normal pressure integral over Σ, only values of ϕ_1 at points on and near to Σ will be required in this work.

Thus the only required values of $\phi_1(r,\theta,s,t)$ are those for which

(1) $\eta = s-f(t)$ lies in the range, $(0,l)$, where l is the length of the body, (10)

(2) the distance of the point (r,θ,s) from the axis of the body is small and $O(\epsilon)$. (11)

The region of integration includes all points, $(0,0,s',t')$, for which

$$\left.\begin{array}{l}(1)\ \eta'' = s'-f(t') \text{ lies in the range } (0,l), \text{ and}\\ (2)\ t-t' = R = +\{r^2+(s-s')^2\}^{\frac{1}{2}}.\end{array}\right\} \tag{12}$$

In the (s',t')-plane, $t-t' = R \geqslant 0$ represents one branch of a rectangular hyperbola with centre at the point (s,t), and asymptotes $(t-t') = \pm(s-s')$.

Contributions to the potential, $\phi_1(r,\theta,s,t)$, come from those regions of the body's trajectory in the (s',t')-plane which are intercepted by the hyperbola $t-t' = R$ (see Figs. 1 and 2). For subsonic velocities only one region of integration occurs, but for supersonic motions there are usually two regions of integration, except when the point $P_2(r,\theta,s,t)$, at which the potential is required, lies near to the body both in space and time; in this case the two regions coalesce (see Fig. 2). Except for transonic velocities, the region of integration is $O(l)$.

If $\boldsymbol{\nu}$ is the outward normal to Σ and $r = \bar{r}(\eta)$ is the equation to the body surface, then from (9), the boundary condition gives

$$\left[\frac{\partial\phi_1}{\partial r}\right]_{r=\bar{r}} = -f'(t)\frac{d\bar{r}}{d\eta}\{1+O(\epsilon^2)\} = \int\limits_G \left\{\frac{q_1(s',t')}{R^3}\right\}_{r=\bar{r}} \bar{r}\,ds'+O(\epsilon^2). \tag{13}$$

On assuming that $q_1(s',t')$ satisfies the Lipschitz condition,

$$|q_1(s,t)-q_1(s_1',t')| < A'|s-s'|+B'|t-t'|, \tag{14}$$

where A' and B' are constants of $O(\epsilon^2)$, we have

$$f'(t)\frac{d\bar{r}}{d\eta} = -\frac{q_1(s,t)}{\bar{r}}\left\{\frac{s-s'}{R}\right\}_G+O(\epsilon^2), \tag{15}$$

since

$$A'\int\limits_G \frac{|s-s'|\bar{r}\,ds'}{R^3} = O(\epsilon^2) \text{ at most;} \tag{16}$$

and a similar result holds for the term containing B'. In this work G consists of one region, only, bounded by the points, $s' = s_3$ and $s' = s_2$. In general $|s_2-s|$ and $|s-s_3|$ are both $O(l)$ and, from (15),

$$f'(t)\bar{r}\frac{d\bar{r}}{d\eta} = 2q_1(s,t)\{1+O(\epsilon)\}. \tag{17}$$

For some points, however, $|s_2-s| = O(\epsilon)$, and/or $|s_3-s| = O(\epsilon)$. In these cases

$$2+O(\epsilon^2) \geqslant \left[\frac{s'-s}{R}\right]_{s_3}^{s_2} \geqslant O(\epsilon^2); \tag{18}$$

the comparative rarity of these points allows of their neglect. If $S(\eta)$ is

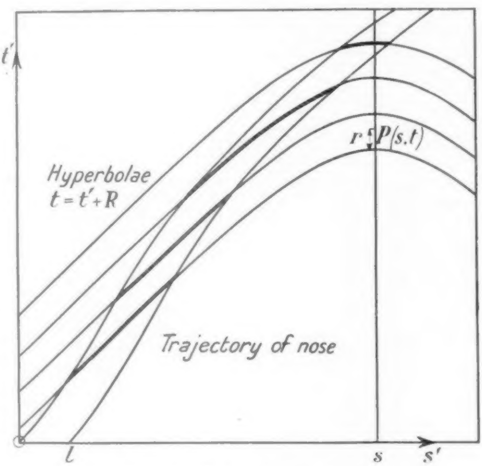

FIG. 1. Subsonic motion.

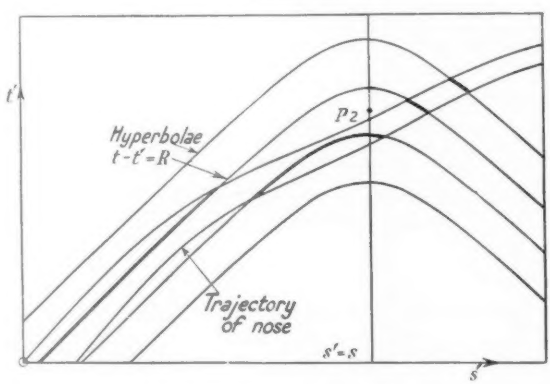

FIG. 2. Supersonic motion.
Thick lines in Figs. 1 and 2 indicate regions of integration.

the area of the cross-section of the body, then the potential appropriate to this motion is

$$\phi_1(r,\theta,s,t) = \int\limits_G \frac{S'(\eta'')f'(t')}{4\pi R}\,ds', \tag{19}$$

where $$\eta'' = s' - f(t'). \tag{20}$$

In the Appendix, ϕ_1 is reduced to the following form,

$$\phi_1(r,\theta,s,t) = \frac{-f'(t)S'(\eta)}{2\pi}\log\tfrac{1}{2}r +$$
$$+\frac{1}{4\pi}\int\limits_H \log(t-t')\frac{\partial}{\partial t'}\{f'(t')(S'(\eta'+t-t')+S'(\eta'+t'-t))\}\,dt', \tag{21}$$

where $\eta' = s-f(t')$, and the region of integration, H, includes all values of t' for which the arguments of S' lie in the range $(0,l)$.

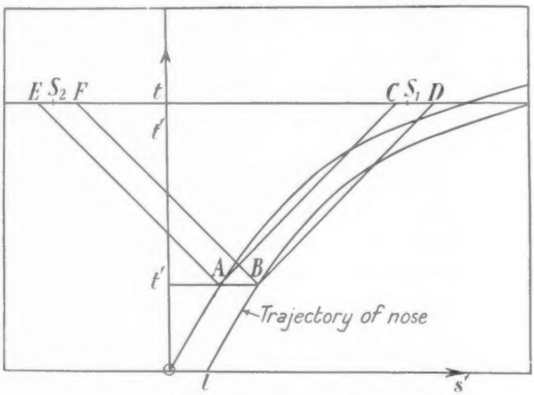

FIG. 3. Influence regions of the body AB.

If $\lambda_1, \lambda_2 = s-f(t')\pm(t-t')$, then the condition that λ_1 should lie in the range $(0,l)$ is equivalent to the condition that S_1 (see Fig. 3) should lie in the domain of influence, CD, of the body AB by waves propagating in the direction s-increasing. A similar result holds for λ_2.

2. *The drag force*

The total inviscid aerodynamic force acting on the body is $\mathbf{F}$, where

$$\mathbf{F} = -\int\limits_\Sigma (p-p_0)\boldsymbol{\nu}\,d\Sigma, \tag{22}$$

and p_0 is the undisturbed pressure.

From Bernoulli's equation the pressure is given by

$$\left(\frac{p-p_0}{\rho_0}\right) = -\frac{\partial\phi_1}{\partial t} - \frac{1}{2}\left(\frac{\partial\phi_1}{\partial r}\right)^2 + O(\epsilon^4\log^2\epsilon), \tag{23}$$

since $$\rho = \rho_0(1+O(\epsilon^2\log\epsilon)). \tag{24}$$

The time derivative $\partial/\partial t$ is taken with respect to axes fixed in space. The drag force, D_0, obtained from (22) and (23), is

$$D_0 = -\mathbf{F}.\mathbf{i} = \frac{\rho_0}{4\pi}\left[-f'^2(t)S'^2(0)\log\left\{\frac{\bar{r}(0)}{2}\right\}-2f''(t)\int\limits_0^l S'^2(\eta)\log\left\{\frac{\bar{r}(\eta)}{2}\right\}d\eta+\right.$$
$$\left.+\int\limits_0^l S'(\eta)\,d\eta\int\limits_H \log(t-t')\frac{\partial\Delta}{\partial t'}\,dt'\right]+O(\epsilon^6\log^2\epsilon), \quad (25)$$

where

$$\Delta = f''(t')\{S'(\psi_1)+S'(\psi_2)\}-\{f'^2(t')+f'(t')f'(t)\}\{S''(\psi_1)+S''(\psi_2)\}, \quad (26)$$

and

$$\psi_1,\psi_2 = s-f(t')\pm(t-t').$$

This expression (25) holds for both sub- and super-sonic motions when the body is pointed at each end. For a body with non-zero base area, however, (25) holds only when the motion is supersonic (and a base drag force, $-p_{\text{base}}\,S(0)$ must be added in this case).

If $lf''(t) \ll f'^2(t)$, and the motion is supersonic, the drag force has the following approximate form:

$$D_0 = \frac{\rho_0}{4\pi}\left[f'^2(t)\left\{-\int\limits_0^l\int\limits_0^l \log|\eta-\sigma|S''(\eta)S''(\sigma)\,d\eta d\sigma-\right.\right.$$
$$\left.-2S'(0)\int\limits_0^l \log(l-\eta)S''(\eta)\,d\eta-S'^2(0)\log\left(\frac{B\bar{r}(0)}{2}\right)\right\}-$$
$$-f''(t)\left\{2\int\limits_0^l S'^2(\eta)\left(\log\frac{B\bar{r}}{2}-\frac{6f'^2(t)}{B}\right)d\eta+\right.$$
$$\left.\left.+2\int\limits_0^l S'(\eta)\,d\eta\int\limits_\eta^l S''(\sigma)\log(\sigma-\eta)\,d\sigma\right\}\right]+O(\epsilon^6\log^2\epsilon), \quad (27)$$

where

$$B^2 = f'^2(t)-1.$$

In practice the effect of acceleration on the drag force will be small compared with the steady velocity effect if the change in the velocity of the body in the time it takes to travel its own length is small compared with the actual velocity of the body.

If the drag force (27) be written in the form

$$D_0 = A_0\,f'^2(t)+B_0\,f''(t), \quad (28)$$

where A_0, B_0 are given by equation (27), then the effect of the acceleration is to make the virtual mass equal to $m+B_0$, where m is the mass of the body and $B_0 = O(\rho_0\,\epsilon^2 V)$, where V is the volume of the body.

B. Motions with oscillations

1. *Introduction*

In this section the motion of the body is taken to be a general oscillation of small amplitude superposed on a variable forward speed. Cylindrical polar coordinates, (r, θ, η), again refer to axes moving with the body, and D/Dt is a time derivative taken with respect to these moving axes. The angle between the geometrical axis of the body and the line of mean motion, and its time derivatives are $O(\epsilon)$. Initially the body is considered to possess an angular velocity, $\alpha'(t)\mathbf{k}$, about an instantaneous axis, $\theta = \frac{1}{2}\pi$, $\eta = \eta_0(t)$; but later an angular velocity, $\beta'(t)\mathbf{j}$, about an axis, $\theta = 0$, $\eta = \eta_2(t)$, is superposed on the above motion to give a general spiral motion. In the former case the velocity of any point of the body is

$$\mathbf{U} = f'(t)\mathbf{i} + g'(t)\mathbf{j}, \tag{29}$$

where

$$g'(t) = \alpha'(t)(\eta - \eta_0(t)) - \alpha(t)f'(t). \tag{30}$$

The boundary condition on Σ gives

$$\mathbf{u}.\boldsymbol{\nu} = \left\{\frac{\partial(\phi_1+\phi_2)}{\partial r}\right\}_{r=\bar{r}} = -f'(t)\frac{d\bar{r}}{d\eta} - g'(t)\cos\theta, \tag{31}$$

where ϕ_2 is the potential due to the oscillatory motion. This boundary condition is satisfied by taking ϕ_2 to be the potential of a line distribution of doublets oriented in the $\mathbf{j}$-direction; thus

$$\phi_2 = \int\limits_G \frac{d}{dx}\left(\frac{q_2(s', t')}{R}\right) ds'. \tag{32}$$

For values of r of order ϵ this doublet potential may be reduced to the following form,

$$\phi_2 = 2q_2(s, t)\frac{\cos\theta}{r} + O(\epsilon^3). \tag{33}$$

The value of $q_2(s, t)$ is now obtained from the boundary condition (31); thus

$$\phi_2 = -g'(t)S(\eta)\frac{\cos\theta}{\pi r}. \tag{34}$$

2. *The aerodynamic forces*

If $\mathbf{R}_0$ is a position vector relative to the instantaneous axis of rotation, then Bernoulli's equation gives

$$\left(\frac{p-p_0}{\rho_0}\right) = -\frac{D\phi}{Dt} - \tfrac{1}{2}(\nabla\phi)^2 + \mathbf{U}_0.\nabla\phi + (\mathbf{R}_0 \wedge \nabla\phi).\alpha'(t)\mathbf{k} + O(\epsilon^4 \log^2\epsilon) \tag{35}$$

(∇ refers to axes fixed in space and $\mathbf{U}_0 = f'(t)\mathbf{i} - \alpha(t)f'(t)\mathbf{j}$).

The drag force, D_α, is obtained from a reduced form of the pressure distribution, (35), thus $D_\alpha = -\mathbf{F}.\mathbf{i}_1$ is given by

$$\left(\frac{D_\alpha - D_0}{\rho_0}\right) = \tfrac{1}{2}S(0)\{\alpha^2(t)f'^2(t) - \alpha'^2(t)\eta_0^2(t)\} + V\{\alpha(t)\alpha'(t)(2f'(t)+\eta_0'(t)) + (\eta_0 - \bar{\eta}_g)(\alpha'^2(t)+\alpha(t)\alpha''(t)) + \alpha^2(t)f''(t)\} + O(\epsilon^6 \log^2\epsilon), \quad (36)$$

where

$$\bar{\eta}_g V = \int_0^l \eta S(\eta)\, d\eta. \quad (37)$$

The effect of linear and angular acceleration on the drag force will be small compared with the steady velocity effect provided that

$$lf''(t) \ll f'^2(t), \quad \text{and} \quad l^2\alpha''(t) \ll \alpha(t)f'^2(t) \quad (38)$$

respectively; also the effect of angular velocity will be small provided that $l\alpha'(t) \ll \alpha(t)f'(t)$. It is interesting to observe that larger forward velocities allow of larger angular velocities before the latter become effectual in the drag force.

The lift force in the plane $\theta = 0$ is denoted by L_x, where, from (22), (34), and (35), we have

$$L_x = \mathbf{F}.\mathbf{j}_1 = \rho_0[V\{\alpha'(t)(f'(t)+\eta_0'(t)) + \alpha''(t)(\eta_0(t) - \bar{\eta}_g) + \alpha(t)f''(t)\} + f'(t)S(0)\{\alpha(t)f'(t) + \alpha'(t)\eta_0(t)\}] + O(\epsilon^5 \log^2\epsilon). \quad (39)$$

Let $\Psi(t)$ be the angle between the resultant velocity of the nose of the body and the axis of the body, viz.

$$\Psi(t) = \alpha(t) - (l - \eta_0)\frac{\alpha'(t)}{f'(t)}; \quad (40)$$

in terms of $\Psi(t)$ the lift force becomes

$$L_x = \rho_0[V\{\Psi(t)f''(t) + \Psi'(t)f'(t) + (l - \bar{\eta}_g)\alpha''(t)\} + f'(t)S(0)\{l\alpha'(t) + \Psi(t)f'(t)\}]. \quad (41)$$

If conditions (38) are satisfied, then the effect of both linear and angular acceleration on the lift force will be small compared with the steady velocity effect.

Let M_y^0 be the pitching moment about the axis, $\theta = \tfrac{1}{2}\pi$, $\eta = 0$, then

$$M_y^0 = \rho_0 V[\bar{\eta}_g\{\Psi'(t)f'(t) + \Psi(t)f''(t)\} + \Psi(t)f'(t) + \alpha''(t)(l\bar{\eta}_g - \bar{\eta}_g^{(2)}) + (l - \bar{\eta}_g)\alpha'(t)f'(t)], \quad (42)$$

where

$$\bar{\eta}_g^{(2)} V = \int_0^l \eta^2 S(\eta)\, d\eta. \quad (43)$$

The analogous pitching moment about the nose is

$$M_y^l = lL_x - M_y^0. \quad (44)$$

For a general oscillation with angular velocity $\alpha'(t)\mathbf{k} + \beta'(t)\mathbf{j}$, the above

method is still applicable; in this case the potential at points near to the body is

$$\phi_3 = \phi_1 + \frac{S(\eta)}{2\pi r}\{h'(t)\sin\theta - g'(t)\cos\theta\}, \tag{45}$$

where $$h'(t) = \beta(t)f'(t) - \beta'(t)(\eta - \eta_2(t)). \tag{46}$$

The additional lateral force, introduced by this rotation about the x-axis, lies in the **k**-direction and is of magnitude

$$L_y = \mathbf{F}.\mathbf{k} = -\rho_0[V\{\beta'(t)\eta_2'(t) + \beta''(t)(\eta_2(t) - \bar{\eta}_g) + \beta'(t)f'(t) + \beta(t)f''(t)\} + \\ + f'(t)S(0)\{\beta'(t)\eta_2(t) + \beta(t)f'(t)\}]. \tag{47}$$

For a body moving at constant incidence α (β) to its own geometric axis, the aerodynamic forces are given by the above equations with

$$\alpha'(t) \equiv 0 \qquad (\beta'(t) \equiv 0).$$

Part II

SLENDER BODY OF GENERAL CROSS-SECTION

1. *Introduction*

In this second part the preceding work is extended to solve identical problems for the slender body of general cross-section.

The method is still to choose a solution of the wave equation which satisfies the boundary condition on Σ: whilst source and doublet potentials were sufficient for the previous motion, in this problem higher-order multisource potentials will be required.

If (r, θ, η) are a set of cylindrical polar coordinates whose polar axis coincides with the axis of the body, then the potential of a distribution of nth-order multisources oriented in the **j**-direction is

$$\phi_n = \int\limits_G \left(\frac{d}{dx}\right)^n \left\{\frac{q_n(s', t')}{R}\right\} ds', \text{ where } R^2 = (s-s')^2 + r^2. \tag{48}$$

For $r = O(\epsilon)$, this may be reduced to the following form (see (33)),

$$\phi_n = A_{n-1}\frac{\cos\{(n-1)\theta\}}{r^{n-1}}, \tag{49}$$

where $$A_{n-1} = (-)^{n-1}2(n-2)!\,q_n(s,t). \tag{50}$$

The general expression for ϕ_0, the potential at points near to the body, may be reduced to give

$$\phi_0 = a_0\log r + b_0 + \sum_{n=1}^{\infty} \frac{A_n\cos(n\theta + \beta_n)}{r^n}, \tag{51}$$

where β_n is the phase angle giving the orientation of each multisource

distribution; β_n may depend on η. A complex potential w, whose real part is ϕ_0, may be defined by

$$w = \phi_0 + i\psi_0 = a_0 \log z + b_0 + \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \tag{52}$$

where $a_n = A_n e^{i\beta_n}$. Thus, ϕ_0 satisfies Laplace's equation in two dimensions, so that the flow is essentially two-dimensional in any plane of cross-section $\eta =$ constant, and the multisource densities a_n are determined from the solution of a two-dimensional incompressible flow problem in which the velocity at infinity is zero.

The body is taken to have an angular velocity, $\alpha'(t)\mathbf{k} + \beta'(t)\mathbf{j}$, with instantaneous axes as specified in Part I.

The velocity of any point of the body is

$$\mathbf{U} = f'(t)\mathbf{i} + g'(t)\mathbf{j} - h'(t)\mathbf{k}, \tag{53}$$

where $g'(t)$ and $h'(t)$ are given by (30) and (46) respectively. From the boundary condition

$$\mathbf{U} . \boldsymbol{\nu} = \left\{\frac{\partial \phi}{\partial \nu}\right\}_{\Sigma} \tag{54}$$

it is found that the source strength a_0 is still given by

$$2\pi a_0 = -S'(\eta) f'(t). \tag{55}$$

It is convenient to determine the aerodynamic force from a momentum integral rather than from the normal pressure integral directly (see (22)). For this we consider a right circular cylindrical control surface, S, surrounding the body and fixed relative to it. The control surface consists of a curved surface S_2, of radius $r_2 = O(\epsilon)$, and circular ends S_1, S_3, lying in the planes $\eta = l$, $\eta = 0$, respectively.

If D/Dt is a time derivative with respect to axes moving with the body, and V_1 is the volume of fluid lying between S and Σ, then the normal pressure integral (22) may be transformed to the following equation,

$$\mathbf{F} = -\int_S \{(p - p_0) I + \rho \mathbf{u}(\mathbf{u} - \mathbf{U})\} . \mathbf{n}\, dS - \int_{V_1} \frac{D(\rho \mathbf{u})}{Dt}\, dv, \tag{56}$$

where $\mathbf{n}$ is a unit vector normal to S, and I is the idemfactor or unit dyadic.

The pressure, obtained from Bernoulli's equation, is given by

$$\left(\frac{p - p_0}{\rho_0}\right) = -\frac{D\phi}{Dt} - \frac{1}{2}(\nabla\phi)^2 + f'(t)\frac{\partial \phi}{\partial \eta} + g'(t)\frac{\partial \phi}{\partial x} + O(\epsilon^4 \log^2 \epsilon). \tag{57}$$

2. *The drag force*

The drag force D is minus the component of $\mathbf{F}$ along the direction of mean motion, $\mathbf{i}_1$; from (56) and (57) we have

$$\frac{D}{\rho_0} = \int\limits_{S_1-S_3} \left\{g'(t)\frac{\partial\phi_0}{\partial x} - \frac{D\phi_0}{Dt} - \frac{1}{2}(\nabla\phi)^2\right\} dS + \int\limits_{V_1} \frac{D}{Dt}\left(\frac{\partial\phi_0}{\partial\eta}\right) dv +$$
$$+ \int\limits_{S_2} \frac{\partial\phi_0}{\partial\eta}\left(\frac{\partial\phi_0}{\partial r} - g'(t)\cos\theta\right) dS_2 + \alpha L_x + O(\epsilon^6\log^2\epsilon), \tag{58}$$

where L_x is the lift force defined later by (78).

The several terms of (58) will be reduced separately. Let C_0 be the contour of the body in the plane $\eta = 0$, then from Gauss's theorem and the equation for the potential, (52), we have

$$\frac{1}{2}\int\limits_{S_1-S_3} (\nabla\phi_0)^2\, dS = \frac{1}{2}\oint\limits_{C_0} \phi_0\frac{\partial\phi_0}{\partial\nu}\, d\tau - \pi\{a_0 b_0\}_{\eta=0} - G(r_2), \tag{59}$$

where $G(r_2)$ is a function of r_2 and the a_n's (see (61)). The divergence theorem gives

$$\mathbf{i}.\int\limits_{V_1} \frac{D}{Dt}(\nabla\phi_0)\, dv = \int\limits_{S_1-S_3} \frac{D\phi_0}{Dt}\, dS - \int\limits_{\Sigma} \frac{D\phi_0}{Dt}(\boldsymbol{\nu}.\mathbf{i})\, d\Sigma. \tag{60}$$

The integral over S_2 may be determined explicitly from the equation for ϕ_0; thus

$$\int\limits_{S_2} \frac{\partial\phi_0}{\partial\eta}\left\{\frac{\partial\phi_0}{\partial r} - g'(t)\cos\theta\right\} dS_2$$
$$= 2\pi\int\limits_0^l a_0\frac{\partial b_0}{\partial\eta}\, d\eta - G(r_2) - \pi[a_1 g'(t)]_{\eta=0}^{\eta=l} + \pi\int\limits_0^l a_1\,\alpha'(t)\, d\eta. \tag{61}$$

Let W be the area in the plane η = constant bounded by the circle, $r = r_2$, and the body contour, $r = \bar{r}(\eta,\theta)$, then let

$$I(\eta) = \int\limits_{W} \left\{\frac{\partial\phi_0}{\partial x} + i\frac{\partial\phi_0}{\partial y}\right\} dW, \tag{62}$$

and so
$$\int\limits_{S_1-S_3} \frac{\partial\phi_0}{\partial x}\, dS = \mathrm{re}\{I(l) - I(0)\}. \tag{63}$$

On defining C to be the contour of the body in the plane η = constant, and C_1 to be the circle $r = r_1$ in the same plane, we may apply the two-dimensional form of Stokes's theorem to give

$$I(\eta) = -i\left\{\oint\limits_{C_1} \phi_0\, dz - \oint\limits_{C} \phi_0\, dz\right\} \quad \{i = \sqrt{(-1)}\}; \tag{64}$$

from the complex potential defined by (52), we have

$$I(\eta) = \pi a_1 + i \oint_C w\,dz + \oint_C \psi_0\,dz. \tag{65}$$

Since w is analytic in the z-plane cut from the origin to $\infty e^{i\alpha}$ and all the singularities of w are enclosed by C, then the series for w converges in the neighbourhood of some point $z = z_0$ on C. If C_r is another closed contour having z_0 in common with C, such that the series for w converges everywhere on C_r, then the second term in (65) is equal to

$$i \int_{C_r} w\,dz = -2\pi a_0 z_0 - 2\pi a_1. \tag{66}$$

On integrating by parts, we have

$$\oint_C \psi_0\,dz = [z\psi_0]_C - \oint z\frac{\partial\psi_0}{\partial\tau}\,d\tau = 2\pi a_0 z_0 \int_C z\frac{\partial\phi_0}{\partial\nu}\,d\tau, \tag{67}$$

and so
$$I(\eta) = -\pi a_1 - \oint_C z\frac{\partial\phi_0}{\partial\nu}\,d\tau. \tag{68}$$

If $z = z_g(\eta)$ is the centre of area of the cross-section $S(\eta)$, then

$$\oint_C f'(t)(\boldsymbol{\nu}.\mathbf{i})\,d\tau = -f'(t)\frac{d}{d\eta}\{S(\eta)z_g(\eta)\}. \tag{69}$$

Also, from the divergence theorem,

$$\oint_C z\boldsymbol{\nu}.(g'(t)\mathbf{j}+h'(t)\mathbf{k})\,d\tau = \{g'(t)+ih'(t)\}S(\eta). \tag{70}$$

On simplifying (68) by means of (54), (69), and (70), we have

$$I(\eta) = -\pi a_1 + f'(t)\frac{d}{d\eta}\{S(\eta)z_g(\eta)\} - (g'(t)+ih'(t))S(\eta). \tag{71}$$

If $\beta(t) \equiv 0$, the drag force, obtained from (58) et seq., is given by

$$\frac{D}{\rho_0} = -\frac{1}{2}\oint_{C_0} \phi_0\frac{\partial\phi_0}{\partial\nu}\,d\tau - \int_\Sigma \frac{D\phi_0}{Dt}\left\{\frac{\partial\phi_0}{\partial\nu} - g'(t)(\mathbf{j}.\boldsymbol{\nu})\right\}\frac{d\Sigma}{f'(t)} +$$
$$+\pi\int_0^l \left(a_1\alpha'(t) - 2b_0\frac{\partial a_0}{\partial\eta} + 2\alpha(t)\frac{Da_1}{Dt}\right)d\eta - \mathrm{re}\Big[\Big\{g'(t)S(\eta) + 2\pi a_1 -$$
$$-f'(t)\frac{\partial}{\partial\eta}\{S(\eta)z_g(\eta)\}\Big\}(-\alpha'(t)\eta_0(t)) - \pi a_0 b_0 + \alpha(t)f''(t)S(\eta)z_g(\eta)\Big]_{\eta=0} -$$
$$-\alpha V\Omega + O(\epsilon^6\log^2\epsilon), \tag{72}$$

where

$$\Omega = \alpha''(t)\eta_0(t) + \alpha'(t)\eta_0'(t) + \alpha'(t)f'(t) + \alpha(t)f''(t) - \alpha''(t)\bar{\eta}_g. \tag{73}$$

The drag force for an oscillating slender body of revolution is obtained for

$$\pi a_1 = -g'(t)S(\eta) \tag{74}$$

(cf. (36)). For direct motion without oscillation, $\alpha(t) \equiv 0$, and the drag force is D_0 where

$$\frac{D_0}{\rho_0} = -\frac{1}{2}\oint_{C_0} \phi_0 \frac{\partial\phi_0}{\partial\nu}\, d\tau - \int_{\Sigma} \frac{D\phi_0}{Dt}\frac{\partial\phi_0}{\partial\nu}\frac{d\Sigma}{f'(t)} - 2\pi\int_0^l b_0 \frac{\partial a_0}{\partial\eta}\, d\eta - \\ -\pi(a_0 b_0)_{\eta=0} + O(\epsilon^6\log^2\epsilon). \tag{75}$$

The third term of (75) has the alternative form:

$$-\pi\log\tfrac{1}{2}(a_0^2)_{\eta=0} - \frac{f'(t)}{2\pi}\int_0^l S'(\eta)\, d\eta \int_0^t \log(t-t')\frac{\partial}{\partial t'}[f'(t')\{S''(\eta'+(t-t'))+ \\ +S''(\eta'-(t-t'))\}]\, dt'. \tag{76}$$

Contributions to (76) come from regions of t' quite near to t; thus if t' lies in the range $(t-\Xi, t)$, then for $f(t)-f(t') \simeq V_0(t-t')$, we shall have $\Xi \simeq \dfrac{l-\eta}{V_0 \pm 1}$, when $V_0 > 1$; whilst, for $V_0 < 1$,

$$\Xi \simeq \frac{l-\eta}{V_0+1}, \quad \text{or} \quad \frac{\eta}{1-V_0}. \tag{77}$$

3. *The lateral force*

The lift force in the $\mathbf{j}_1$ direction is L_x where, from (56) and (57),

$$L_x = \mathbf{F}.\mathbf{j}_1 \\ = \rho_0 f'(t)\int_{S_1-S_3} \frac{\partial\phi_0}{\partial x}\, dS + \rho_0\int_{S_2}\left\{\left(\frac{p-p_0}{\rho_0}\right)\cos\theta + \frac{\partial\phi_0}{\partial x}\left\{\frac{\partial\phi_0}{\partial r} - g'(t)\cos\theta\right\}\right\} dS - \\ -\rho_0\int_{V_1} \frac{D}{Dt}\left(\frac{\partial\phi_0}{\partial x}\right) dv + O(\epsilon^5\log^2\epsilon). \tag{78}$$

A complex lateral force, L, may be defined by $L = L_x + iL_y$; in the above notation,

$$L = \rho_0\int_0^l \left\{f'(t)\frac{\partial}{\partial\eta} - \frac{D}{Dt}\right\}(I(\eta) - \pi a_1)\, d\eta + O(\epsilon^5\log^2\epsilon). \tag{79}$$

On substituting the value of $I(\eta)$ from (71) into this equation, we have

$$\frac{L}{\rho_0} = f''(t)S(0)z_g(0) - f'^2(t)\{S'(0)z_g(0) + S(0)z_g'(0)\} - \\ -\{\alpha'(t)\eta_0(t) + \alpha(t)f'(t)\}f'(t)S(0) + 2\pi f'(t)[a_1]_{\eta=0} + 2\pi\int^l \frac{Da_1}{Dt}\, d\eta - \\ -V\Omega + O(\epsilon^5\log^2\epsilon). \tag{80}$$

The angular velocity, $\beta'(t)\mathbf{j}$, introduces extra terms, iL_y, into (80), where L_y is given by (47).

A complex pitching moment, M^0, is defined as the moment of the lateral force about the base; (71) and (79) give

$$M^0 = M_x^0 + iM_y^0 = i\rho_0 \int\limits_0^l \eta\, d\eta \left\{ f'(t)\frac{\partial}{\partial \eta} - \frac{D}{Dt} \right\} \{I(\eta) - \tfrac{1}{\pi} a_1\}. \tag{81}$$

If
$$\int\limits_0^l \eta^2 S(\eta)\, d\eta = \bar{\eta}_g^{(2)} V, \quad \text{and} \quad \int\limits_0^l z_g(\eta) S(\eta)\, d\eta = \bar{z}_g V, \tag{82}$$

then

$$M^0 = i\rho_0 \Bigg[2\pi \int\limits_0^l \left\{ f'(t)a_1 + \eta \frac{Da_1}{Dt} \right\} d\eta + f'(t)S(0)z_g(0) + f''(t)\bar{z}_g V +$$
$$+ V[\alpha'(t)f'(t)\eta_0(t) + \alpha(t)f'^2(t) - \alpha''(t)\bar{\eta}_g^{(2)} +$$
$$+ \bar{\eta}_g\{\alpha''(t)\eta_0(t) + \alpha(t)f''(t) + \alpha'(t)\eta_0'(t)\}] \Bigg]. \tag{83}$$

Thus although the drag force depends on all the multisource densities, a_n, the lift force depends only on the first one, a_1.

APPENDIX

Reduction of the Source Potential

In this appendix the source potential (19) is reduced to a more convenient expression, (21).

The source potential is

$$\phi_1(r, \theta, s, t) = \frac{1}{4\pi} \int\limits_G f'(t')S'(\eta'') \frac{ds'}{R}, \tag{a.1}$$

where
$$t' = t - R, \quad \eta'' = s' - f(t') \quad \text{and} \quad R^2 = (s - s')^2 + r^2. \tag{a.2}$$

For points (r, θ, s, t) near to the body both in space and time, G consists of one region only bounded by the limits $s' = s_3$ and $s' = s_2$.

On integration, we have

$$\int\limits_{s_3}^{s_2} \frac{ds'}{R} = -\left[\log\left\{\frac{R + s - s'}{r}\right\}\right]_{s_3}^{s_2}. \tag{a.3}$$

Thus (a. 1) may be integrated by parts to give

$$-4\pi\phi_1 = \left[f'(t')S'(\eta'')\log\left\{\frac{R + s - s'}{2}\right\}\right]_{s_3}^{s_2} - \int\limits_{s_3}^{s_2} \log\left\{\frac{R + s - s'}{2}\right\} \frac{\partial}{\partial s'}\{S'(\eta'')f'(t')\}\, ds'. \tag{a.4}$$

For $s < s'$,
$$\log\left\{\frac{R + s - s'}{2}\right\} = -\log\left\{\frac{R + s' - s}{2}\right\} - \log\left(\frac{4}{r^2}\right), \tag{a.5}$$

and so

$$\int_{s_3}^{s_2} \log\left\{\frac{R+s-s'}{2}\right\}\frac{\partial}{\partial s'}\{S'(\eta'')f'(t')\}\,ds'$$

$$= \left(\int_{s_3}^{s} - \int_{s}^{s_2}\right)\left[\log\left\{\frac{R+|s-s'|}{2}\right\}\frac{\partial}{\partial s'}\{S'(\eta'')f'(t')\}\right]ds' + 2\log(\tfrac{1}{2}r)[S'(\eta'')f'(t')]_{s}^{s_2}. \quad \text{(a. 6)}$$

If the first term in the right-hand side of (a. 6) is denoted by Q, and

$$E = f'(t')S'(\eta'')\log\left\{\frac{R+|s-s'|}{2}\right\}, \quad \text{(a. 7)}$$

then (a. 4) and (a. 6) give

$$4\pi\phi_1 = Q+[E]_{s'=s_3}+[E]_{s'=s_2}-2S'(\eta)f'(t)\log\tfrac{1}{2}r. \quad \text{(a. 8)}$$

At both limits, $s' = s_3$, $s' = s_2$, we have $\eta'' =$ either 0 or l, (see Figs. 1 and 2).

Consider the expression E:

(a) for bodies pointed at each end, E vanishes at both limits since $S'(0) = 0$ and $S'(l) = 0$;

(b) in the supersonic case one region of integration, only, occurs when the point (r, θ, s, t) lies on or near to the body; in this case both limits correspond to $\eta'' = l$ and, since $S'(l) = 0$, E vanishes at both limits even when $S(0) \neq 0$.

It can be shown that

$$\left(\int_{s_3}^{s} - \int_{s}^{s_2}\right)\left[\log\left\{\frac{R+|s-s'|}{2}\right\}\frac{\partial}{\partial s'}\{S'(\eta'')f'(t')\}\right]ds' = O(\epsilon^3), \quad \text{(a. 9)}$$

and so

$$Q = \left(\int_{s_3}^{s} - \int_{s}^{s_2}\right)\left[\log R\frac{\partial}{\partial s'}\{S'(\eta'')f'(t')\}\right]+O(\epsilon^3). \quad \text{(a. 10)}$$

For $|s-s'| = O(l)$, we have $(t-t') = R = |s-s'|+O(\epsilon^2)$, and

$$\eta'' = s-f(t')+s'-s = s-f(t')\pm(t-t')+O(\epsilon^2). \quad \text{(a. 11)}$$

But values of s' for which $|s-s'| = O(\epsilon)$ only contribute an amount of $O(\epsilon^3\log\epsilon)$ to Q and so

$$Q = \int_{H} \log(t-t')\frac{\partial}{\partial t'}\{f'(t')(S'(\eta'+t-t')+S'(\eta'-t+t'))\}\,dt'+O(\epsilon^3\log\epsilon), \quad \text{(a. 12)}$$

where H extends over all values of t' for which the arguments of S' lie in the range $(0, l)$.

From (a. 8) and (a. 12) we have

$$4\pi\phi_1 = -2S'(\eta)f'(t)\log\tfrac{1}{2}r+\int_{H} \log(t-t')\frac{\partial}{\partial t'}\{f'(t')(S'(\eta'+t-t')+$$
$$+S'(\eta'-t+t'))\}\,dt'+O(\epsilon^3\log\epsilon). \quad \text{(a. 13)}$$

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THE ENERGY DISTRIBUTION BEHIND A DECAYING TWO-DIMENSIONAL SHOCK

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SUMMARY

In this paper the work of Lighthill (**1**) for one-dimensional unsteady motion is extended to the problem of steady, supersonic flow past a two-dimensional slender body. Goldsworthy (**2**) and others have shown that, in the limit when the Mach number, M, of the undisturbed flow tends to infinity, the two motions become identical.

As in (**1**), an accuracy hypothesis is formulated concerning the degree of accuracy of the Friedrichs theory (**3**) for nearly plane, two-dimensional, steady shocks. A comprehensive check of this theory on the lines of the accuracy hypothesis is made for a fluid with general thermodynamic properties: in the process of justifying this hypothesis the picture of the flow as a whole is improved.

A relationship is found between the increased total energy flux across a plane transverse to the flow in the residual wave, and the entropy flux across the region of the shock upstream from this plane. The relationship is checked by comparing the value of the pressure fluctuations in the residual wave (due to entropy change at the shock) as determined from a consideration of conditions in the residual wave, with the value obtained from an investigation of the local reflection of the simple wave in the shock.

The work is extended to the problem of the actual flow past a two-dimensional aerofoil when there are two shocks forming an N-wave, and in this case, conditions very far downstream are investigated.

1. Introduction

THE work of Lighthill (**1**) for one-dimensional piston motions is extended to the problem of the two-dimensional, steady, supersonic flow past a thin symmetrical body. Goldsworthy (**2**) has shown that these two motions become identical as the Mach number, M, of the undisturbed (two-dimensional) flow tends to infinity.

For both these motions, Friedrichs (**3**) has put forward theories which neglect the third-order entropy change at the shocks. By assuming an accuracy hypothesis for the Friedrichs theory for one-dimensional motion, Lighthill has been able to sketch into this earlier theory the effect of the third-order entropy change, and then to check the hypothesis.

In this paper a similar accuracy hypothesis is formulated for the Friedrichs theory for steady, two-dimensional flow, and results similar to those derived by Lighthill are obtained. At first the decay of a single nose-shock attached to the body is investigated, but later this work is extended to

the problem of the N-wave. For this two shocks occur, one attached to the nose and one attached to the tail of the body: the region lying between these two shocks is a simple wave.

On the Friedrichs theory the flow behind a single shock is a simple wave followed by an undisturbed region—the residual wave. Errors in this theory are chiefly due to: (1) neglecting the third-order entropy change at the shock, (2) applying a second-order boundary condition at the shock. These errors suggest the following:

Accuracy hypothesis

If the simple wave is altered in the following manner, then the changes produced are of the same order as the errors arising in the Friedrichs theory:

(1) the specific entropy of the whole wave is altered to its known maximum,

(2) the slope of the body surface is altered continuously by an amount of order the cube of the maximum shock strength.

If this hypothesis is valid the Friedrichs theory will be of considerable value at any distance downstream. The theory is checked by comparing the value of the pressure fluctuations in the residual wave, as determined from a consideration of conditions in this region, with the value obtained from an investigation of the local reflection of the simple wave in the shock. The work is modified for the N-wave, and in this case conditions well downstream of the body are investigated.

A brief résumé only of the work is given, since the analysis is similar to the work of Lighthill (**1**).

2. The entropy-energy flux relation

The notation for the physical quantities used in this paper is as follows:

Quantity	*Undisturbed flow*	*Disturbed flow*
Pressure	P	$P+p$
Specific volume	V	$V-v$
Specific entropy	0	s
Fluid velocity	$(U, 0)$	$(U+u, w)$
Flow angle	0	$\theta = \dfrac{w}{U+u}+O(\theta^3)$
Sonic velocity	$a_0 = V\sqrt{P_V}$	$a = (V-v)\sqrt{\left(\dfrac{\partial p}{\partial v}\right)_s}$
Temperature	T	—

In the undisturbed flow, derivatives will be indicated with capital letters as suffixes; thus P_V is the derivative of the pressure with respect to minus the volume at constant entropy. The usual partial derivatives will be used for the disturbed flow, i.e. the afore-mentioned derivative is $(\partial p/\partial v)_s$.

The steady, two-dimensional, isentropic equations of fluid motion may be solved to give values for the fluid perturbation velocity components in the region of the simple wave; these are

$$\left.\begin{aligned} w &= U\{\theta-\theta^2(M^2-1)^{-\frac{1}{2}}+O(\theta^3)\} \\ u &= U\{-\theta(M^2-1)^{-\frac{1}{2}}+\theta^2(M^2-1)^{-\frac{1}{2}}\}\left\{1-\frac{M^4VP_{VV}}{4(M^2-1)P_V}\right\}+O(\theta^3) \end{aligned}\right\}. \tag{1}$$

The flow behind the shock is a simple wave in which the flow quantities are constant on straight characteristics inclined at an angle ($\tan^{-1}\zeta_+$) with the axis of the body. If $\xi(\theta)$ is the intercept of a ζ_+ characteristic on the y-axis (see Figs. 1 and 2), then these characteristics have equation $y = \xi(\theta)+x\zeta_+$, where ζ_+ is obtained to second order in θ from simple wave theory.

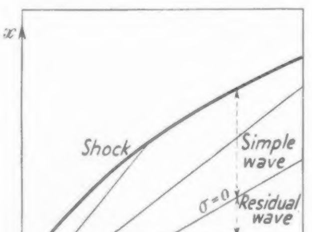

FIG. 1. One-dimensional unsteady motion.

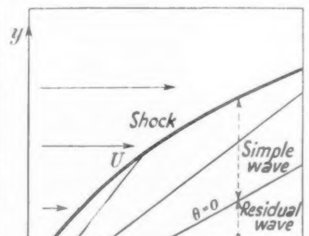

FIG. 2. Two-dimensional steady flow.

The specific entropy gain at the shock is determined from the shock conditions. Also, the rate at which the body is doing work in moving through the fluid, and the flux of increased total energy across that part of the plane x = constant which lies in the simple wave (see Fig. 2) are both determined, to second order in θ, from the simple wave theory. The difference between these two expressions (a second-order quantity) must represent increased total energy flux across the residual wave; the latter is denoted by $R(x)$. The form of $R(x)$ suggests that it depends only on shock discontinuities upstream from the plane x = constant; if $S(x)$ is the total entropy flux across this region of the shock, then it is found that

$$R(x) = TS(x)\left\{1-\frac{T_V P}{2TP_V}-\frac{PP_{VV}M^2}{8P_V^2(M^2-1)}\right\}; \tag{2}$$

for a perfect gas
$$\frac{R(x)}{TS(x)} = \frac{(3M^2-4)(\gamma+1)}{8\gamma(M^2-1)}, \tag{3}$$

where γ is the ratio of the specific heats.

Equation (2) is identical with the result found by Lighthill (corresponding to the case $M = \infty$) except for the factor $M^2/(M^2-1)$.

3. The pressure pulse check

On the accuracy hypothesis the velocities and thermodynamic variables are of third order in the residual wave. The pressure fluctuations in this region satisfy the wave equation to a first approximation; and so a solution for p under the boundary condition, $[dp/dy]_\Sigma = 0$, is

$$p = F\{x+y\sqrt{(M^2-1)}\}+F\{x-y\sqrt{(M^2-1)}\}. \tag{4}$$

Moreover, the explicit value of p may be obtained from relationships existing between the third-order velocities and thermodynamic variables in the residual wave; thus

$$p = \frac{VTM}{16a_0\sqrt{(M^2-1)}}\left\{\frac{4T_V}{T}-\frac{P_{VV}M^2}{P_V(M^2-1)}+\frac{8}{M^2V}\right\}\times$$
$$\times[S'\{\tfrac{1}{2}(x+y\sqrt{(M^2-1)})\}+S'\{\tfrac{1}{2}(x-y\sqrt{(M^2-1)})\}]. \tag{5}$$

The pressure variations in the residual wave are due to a pressure pulse propagating from the shock along ζ-characteristics and of strength

$$\frac{T}{16U(M^2-1)^{\frac{3}{2}}}\{(3\gamma-5)M^4-4(\gamma-3)M^2-8\}\frac{dS(x)}{dx}, \tag{6}$$

for a perfect gas. The first term in the square brackets in (5) represents a pressure pulse which may be considered as a reflection of the simple wave in the shock; this pulse is propagated along characteristics with slope $dy/dx = \zeta_-$ (< 0). The second term in (5) represents the reflection of this reflected wave, at the body surface, and is propagated along ζ_+ characteristics.

Consider the reflection of the simple wave at the shock. The reflected pressure wave, determined from a study of local conditions just behind the shock, is propagated unchanged (to a first approximation) through the simple wave and into the residual wave. The theory is checked by comparing this reflected pressure wave with its value given by (5).

The pressure gradients along the two families of characteristics are obtained from a knowledge of the shock discontinuities and the exact characteristic equations; these are

$$\frac{d\theta}{dp} = \pm\frac{(W^2-a^2)^{\frac{1}{2}}(V-v)}{aW^2}, \text{ on } \frac{dy}{dx} = \zeta_\pm, \tag{7}$$

where

$$W^2 = (U+u)^2+w^2 \quad \text{(see 4)}. \tag{8}$$

A reflection coefficient, r, is defined as the ratio of the pressure gradients across the simple wave and its reflection at the shock; thus we find

$$r = \frac{M^4 P_{VV}^2}{128 V P_V^2 (M^2-1)^{7/2}} \left\{ \frac{T_V V}{T} - \frac{M^2 V P_{VV}}{P_V(M^2-1)} + \frac{2}{M^2} \right\} v^3, \tag{9}$$

where v is the specific volume perturbation.

From this value of r and the value of $[dp/dx]$ along ζ_+ characteristics, obtained from simple wave theory, the equation for the pressure pulse along ζ_- characteristics is obtained: it is, in fact (6).

The pressure pulses reflected at the body surface gradually slow down the characteristics in the residual wave.

4. The N-wave

The previous work for a single shock may be extended to the case when the body is finite in length, so that two shocks occur, one at the nose and one at the tail of the body. Both shocks decay in strength as they recede from the body whilst the flow between them is taken to be a simple wave.

On the Friedrichs theory, the fluid behind the rear shock is undisturbed to second order, and the rear shock conditions are satisfied to this order. Since conditions behind the rear shock are known with an error of order the cube of the maximum shock strength, this error may also be incurred in the flow quantities in the simple wave. Thus, at large distances downstream, the motion of the rear shock may be given quite inaccurately by simple wave theory.

The work is developed in a manner similar to that outlined above for a single shock. The total increased energy flux across that part of the plane $x =$ constant which lies in the residual wave, and the entropy flux across the regions of both shocks lying upstream from this plane, are both determined explicitly, and relation (2) is found to hold between them. This work is checked by a pressure pulse method. The pressure pulse is still given by (5), although, because of the new value of the entropy flux, the pressure pulse becomes

$$p = \frac{M^8 P_{VV} V^2}{192(M^2-1)^{5/2}} \left\{ \frac{4T_V V}{T} - \frac{M^2 V P_{VV}}{P_V(M^2-1)} + \frac{8}{M^2} \right\} (\theta_1^3 - \theta_2^3), \tag{10}$$

where θ_1, θ_2 are the values of θ at the front and rear shocks respectively. In the latter equation the term in θ_1 represents the reflection of the simple wave at the shock, whilst the transmission of this reflected wave through the rear shock gives rise to the term in θ_2.

At large distances downstream, the equation to the rear shock is unreliable, for the reflected pressure waves will gradually overtake the rear shock and the equation given by the Friedrichs theory will be modified.

Very far downstream the energy flux is divided between the following modes:

(*a*) energy transferred to infinity in the N-wave,

(*b*) energy flux across the residual wave associated with entropy flux across the shocks,

(*c*) energy flux across the residual wave due to the pressure wave behind the rear shock.

Far downstream the ratios of the energy fluxes in these three modes become

$$\left(\frac{T_V P}{2TP_V}+\frac{M^2 P P_{VV}}{8P_V^2(M^2-1)}\right); \qquad \left(1-\frac{T_V P}{TP_V}-\frac{P}{M^2 V P_V}\right);$$
$$\left(\frac{PT_V}{2P_V T}-\frac{PP_{VV}M^2}{8P_V^2(M^2-1)}+\frac{P}{M^2 V P_V}\right). \tag{11}$$

The total energy flux is $TS(\infty)$ in this case.

This is not the final state, however, for ultimately the energy associated with the pressure wave, (*c*), is gradually transferred to the N-wave, (*a*), and all the energy which has not gone into increasing the entropy of the fluid without pressure change is in progress to infinity in the N-wave. This final energy transfer occurs at a distance downstream of $O(x_3/\theta_0)$, where x_3 is the length of the body and θ_0 is the value of θ at the nose. When the body is not symmetrical, the energy flux in the N-wave dominates the energy fluxes in the other modes.

The author wishes to thank Mr. G. N. Ward for his help and encouragement and Professor Lighthill, who suggested the problem, for his criticism and advice.

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THE DIFFRACTION OF SOUND PULSES BY AN OSCILLATING, INFINITELY LONG STRIP

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SUMMARY

In this paper the solution is obtained for the problem of a plane pressure pulse normally incident on an infinitely long strip of finite width, capable of motion as a spring-supported rigid body.

Numerical calculations have been performed for the case when the incident pulse is sharp-fronted and of constant unit pressure, and graphs drawn to illustrate the way in which the mobility of the strip affects the pressure distribution on the back of it.

1. Introduction

THE problem considered is the two-dimensional one of an infinitely long strip of finite width subjected to a plane-fronted sound pulse at normal incidence. The strip may be regarded as the obstacle produced when a wall perpendicular to the ground is 'reflected' in the plane of the ground.

For an incident sinusoidal wave-train there are the classical solutions of Rayleigh (**1**) for the extreme case of wave-length large compared with the dimensions of the obstacle. Other more recent investigators in this field include Copson (**2**), Sieger (**3**), and Morse and Rubinstein (**4**). In 1948 E. N. Fox (**5**) obtained a solution of the stationary strip problem with an incident pulse of a general two-dimensional form, and carried out computations in the particular case of a sharp-fronted pulse of constant unit pressure.

In this paper, based on this solution by Fox, the two-dimensional problem is investigated when the strip is capable of motion as a spring-supported, rigid body. By a general principle propounded in Fox's paper, to find the pressure at any point it is sufficient to obtain the pressure on the back of the strip, since these two quantities are directly related.

Adapting results obtained by Fox, the exact solution for the pressure on the back of the strip has been obtained for all moments up to a time $4l/a$ after the onset of the pulse, where $2l$ is the width of the strip and a is the velocity of sound. Calculations have been carried out on the pressure on the back of the strip up to a time $2l/a$, and on the velocity of the strip up to time $4l/a$.

No calculations have been carried out on the pressure at points off the strip, but, as Fox points out, it is doubtful whether such calculation would yield any new results of appreciable physical interest. In effect, points on the back of the strip are in the deepest 'shadow', and thus the results obtained indicate the most pronounced effects of diffraction to be expected in the problem.

2. Strip moving as a rigid body

Let us now consider the problem in which an infinite strip of finite width is struck at normal incidence by a plane-fronted pressure pulse. For simplicity in writing formulae we make the system non-dimensional by taking the width, $2l$, of the strip as the unit of length and the time, $2l/a$, taken by a sound wave in travelling this distance as the unit of time. The wave velocity is then unity and we may write the incident pressure pulse as $p_0(t+y)$, where time is measured from the instant at which the pulse strikes the strip.

In the (x, y)-plane the strip is represented by the straight line $y = 0$, $0 \leqslant x \leqslant 1$ when at rest. The strip is considered as being a rigid body, able to move in one-dimensional motion parallel to itself, as if supported by a spring with damping.

Let the total pressure set up be $p_T(x, y, t)$ and write

$$p_T(x,y,t) = p_0(t+y)+p(x,y,t) \tag{1}$$

where $p(x, y, t)$ is the diffracted pressure.

Taking the corresponding velocity potential as ϕ, and the velocity of the strip at any moment as $U(t)$, then for a point on the strip we have

$$\partial\phi/\partial y = U(t);$$

or, as we are assuming small oscillations,

$$\left[\frac{\partial\phi}{\partial y}\right]_{y=0} = U(t) \text{ approximately.} \tag{2}$$

Hence, eliminating ϕ between (2) and the equation $p_T = -\rho\,\partial\phi/\partial t$, where ρ is the density, and substituting from (1), we find

$$\left[\frac{\partial p}{\partial y}\right]_{y=0} = -\rho\frac{dU}{dt}-\frac{dp_0}{dt}. \tag{2a}$$

Now let us compare this with the postulate of Fox's stationary strip problem, when the incident pressure is $[\rho U(t+y)+p_0(t+y)]$. Denoting the total and diffracted pressures thus obtained by $p_T^{(1)}(x, y, t)$ and $p^{(1)}(x, y, t)$ respectively, then the appropriate boundary condition is that

$$\left[\frac{\partial p_T^{(1)}}{\partial y}\right]_{y=0} = 0, \qquad \text{i.e. } \left[\frac{\partial p^{(1)}}{\partial y}\right]_{y=0} = -\rho\frac{dU}{dt}-\frac{dp_0}{dt}.$$

Thus we see that the present problem and that of Fox are mathematically identical, and we may write

$$p(x,y,t) = p^{(1)}(x,y,t),$$

or,

$$p_T(x,y,t) = p_T^{(1)}(x,y,t) - \rho U(t+y). \tag{3}$$

2.1. *First interval:* $0 \leqslant t \leqslant 1$

Fox showed that in the interval $0 \leqslant t \leqslant 1$ the contribution to $p_T^{(1)}(x,-0,t)$ from the edge $x = 0$ is

$$p_1(x,t) = \frac{1}{\pi} \int\limits_0^{t-x} \frac{\rho U(z)+p_0(z)}{t-z} \sqrt{\left(\frac{x}{t-x-z}\right)}\, dz\, H(t-x), \tag{4}$$

where

$$H(t-x) = 1 \quad \text{if } t > x \tag{4 a}$$
$$= 0 \quad \text{if } t < x.$$

The contribution from the edge $x = 1$ is

$$p_2(x,t) = p_1[(1-x),t]. \tag{5}$$

Thus, in the interval $0 \leqslant t \leqslant 1$,

$$p_T(x,-0,t) - p_T(x,+0,t) = 2[p_1+p_2-\rho U(t)-p_0(t)]. \tag{6}$$

We can now formulate the equation for $\rho U(t)$, since, for the motion of the strip, we have the equation

$$m\frac{dU}{dt} + cU + k\int\limits_0^t U(z)\,dz = \int\limits_0^1 [p_T(x,-0,t)-p_T(x,+0,t)]\,dx, \tag{7}$$

where m, c, and k are constants connected with the mass, damping, and spring stiffness involved.

On substituting for the integral on the right-hand side of (7) from (6), it is found that the integration can be carried out, after changing the order of integration, and thence we find

$$m\frac{dU}{dt} + (c+2\rho)U + (k-2\rho)\int\limits_0^t U(z)\,dz = 2\left[\int\limits_0^t p_0(z)\,dz - p_0(t)\right]. \tag{8}$$

This is the equation of a forced damped oscillation, the precise nature of which will be determined by the relations between ρ and the constants m, c, and k.

Assuming that $U(0) = 0$ and that the roots $q = \alpha$ and $q = \beta$ of the equation

$$mq^2 + (c+2\rho)q + k - 2\rho = 0$$

are distinct, equation (8) can be solved to give

$$U(t) = \frac{2}{m}\int\limits_0^t p_0(t-z)\left[\frac{1-\alpha}{\alpha-\beta}e^{\alpha z} - \frac{1-\beta}{\alpha-\beta}e^{\beta z}\right] dz. \tag{9}$$

The pressure on the back of the strip is found by inserting this expression for $U(t)$ in (4) and (5) and applying (6).

2.2. *Second interval:* $1 \leqslant t \leqslant 2$.

Fox showed that in the interval $1 \leqslant t \leqslant 2$ the contribution from the edge $x = 0$ to $p_T^{(1)}(x, -0, t)$ is

$$p_3(x,t) = p_1(x,t) - F(x,t), \tag{10}$$

where
$$F(x,t) = \frac{1}{\pi} \int_0^\infty \frac{p_1(1+y, t-x-2y)}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{2}} dy,$$

i.e.

$$F(x,t) = \frac{1}{\pi^2} \int_0^{(T-x)/2} \frac{1}{x+y} \left(\frac{x}{y}\right)^{\frac{1}{2}} dy \int_0^{T-x-2y} \frac{\rho U(z)+p_0(z)}{1+T-x-y-z} \left(\frac{1+y}{T-x-2y-z}\right)^{\frac{1}{2}} dz\, H(T-x), \tag{11}$$

and $T = (t-1)$.

The contribution from the edge $x = 1$ to $p_T^{(1)}(x, -0, t)$ is

$$p_4(x,t) = p_3[(1-x), t]. \tag{12}$$

In the course of the solution we shall require $\int_0^1 p_3\, dx$; let us first evaluate $\int_0^1 p_1\, dx$. If we write

$$w(z) = \rho U(z) + p_0(z), \tag{13}$$

then, after performing a change of order of integration and evaluating the inner integral so produced, we find

$$\int_0^1 p_1(x,t)\, dx = \frac{1}{2} \int_0^t w(z)\, dz - \frac{1}{\pi} \int_0^T w(z) \left[\tan^{-1}(T-z)^{\frac{1}{2}} + \frac{(T-z)^{\frac{1}{2}}}{1+T-z}\right] dz. \tag{14}$$

Dealing in a similar manner with $\int_0^1 F(x,t)\, dx$ we find

$$\int_0^1 F(x,t)\, dx = \frac{1}{\pi} \int_0^T w(z) \left[\tan^{-1}(T-z)^{\frac{1}{2}} + \frac{(T-z)^{\frac{1}{2}}}{1+T-z}\right] dz. \tag{15}$$

Hence,

$$\int_0^1 [p_T(x,-0,t) - p_T(x,+0,t)]\, dx$$

$$= 2\left\{\int_0^t w(z)\, dz - w(t) - -\frac{4}{\pi} \int_0^T w(z) \left[\tan^{-1}(T-z)^{\frac{1}{2}} + \frac{(T-z)^{\frac{1}{2}}}{T+1-z}\right] dz\right\},$$

and this can now be substituted in the equation of motion. If at the same time we put

$$U_1(t) = U(t) - \frac{2}{m}\int\limits_0^t p_0(t-z)\left[\frac{1-\alpha}{\alpha-\beta}e^{\alpha z} - \frac{1-\beta}{\alpha-\beta}e^{\beta z}\right] dz,$$

we find that the equation satisfied by $U_1(t)$ is

$$m\frac{dU_1}{dt} + (c+2\rho)U_1 + (k-2\rho)\int\limits_0^t U_1(z)\, dz$$

$$= -\frac{8}{\pi}\int\limits_0^{t-1} [\rho U(z) + p_0(z)]\left[\tan^{-1}(t-1-z)^{\frac{1}{2}} + \frac{(t-1-z)^{\frac{1}{2}}}{t-z}\right] dz, \quad (16)$$

where the right-hand side is known, since it involves $U(z)$ for $0 \leqslant z \leqslant 1$ only.

This can be solved by Heaviside operational methods, and we find

$$U(t) = \frac{2}{m}\int\limits_0^t p_0(t-z)\left[\frac{1-\alpha}{\alpha-\beta}e^{\alpha z} - \frac{1-\beta}{\alpha-\beta}e^{\beta z}\right] dz -$$

$$-\frac{8}{m\pi(\alpha-\beta)}\int\limits_0^{t-1}\left[\frac{y^{\frac{1}{2}}}{1+y} + \tan^{-1}y^{\frac{1}{2}}\right] dy \int\limits_0^{t-y-1} p_0(t-y-1-z)[\alpha e^{\alpha z} - \beta e^{\beta z}]\, dz -$$

$$-\frac{16\rho}{m^2\pi(\alpha-\beta)^2}\int\limits_0^{t-1}\left[\frac{y^{\frac{1}{2}}}{1+y} + \tan^{-1}y^{\frac{1}{2}}\right] dy \int\limits_0^{t-y-1} p_0(t-y-1-z) \times$$

$$\times\left\{\alpha(1-\alpha)ze^{\alpha z} + \beta(1-\beta)ze^{\beta z} + \frac{2\alpha\beta-\alpha-\beta}{\alpha-\beta}[e^{\alpha z} - e^{\beta z}]\right\} dz. \quad (17)$$

3. Numerical calculations

Calculations were carried out for the case of an incident pressure pulse $H(t+y)$, where $H(t)$ is defined as in (4 a). Also c and k have been taken as zero, to simplify calculations, justification for this lying in the fact that, from mechanical considerations, the inertia effect will be predominant in the early stages of such an oscillation.

During the period $0 \leqslant t \leqslant 1$ the velocity and the pressure on the back of the strip for $x = 0{\cdot}1$, $0{\cdot}3$, and $0{\cdot}5$ have been calculated at intervals of $0{\cdot}1$ in time. During the period $1 \leqslant t \leqslant 2$ the velocity, only, has been calculated at intervals of $0{\cdot}2$ in time. The expression for the pressure on the back of the strip was complicated and any calculation on it would hardly have been justified.

The interval $0 \leqslant t \leqslant 1$

Taking $c = k = 0$ and $p_0(z) = H(z)$ we find

$$U(t) = -\frac{1}{\rho}+\frac{2}{m(\alpha-\beta)}\left[\frac{1-\alpha}{\alpha}e^{\alpha t}-\frac{1-\beta}{\beta}e^{\beta t}\right], \tag{18}$$

where
$$\alpha = \frac{-\rho+(\rho^2+2\rho m)^{\frac{1}{2}}}{m}, \qquad \beta = \frac{-\rho-(\rho^2+2\rho m)^{\frac{1}{2}}}{m},$$

and
$$\rho = 8l^3\times(\text{density of air}),$$
$$m = 4l^2\times(\text{thickness of wall})\times(\text{density of wall}).$$

Also
$$p_1 = \frac{H(t-x)}{\pi(\alpha-\beta)}\,\frac{2\rho}{m}\left[\frac{1-\alpha}{\alpha}I(\alpha)-\frac{1-\beta}{\beta}I(\beta)\right], \tag{19}$$

where
$$I(\alpha) = 2e^{\alpha(t-x)}x^{\frac{1}{2}}\int\limits_0^{\sqrt{(t-x)}}\frac{e^{-\alpha u^2}}{u^2+x}\,du, \tag{20}$$

this being in the form computed.

Calculations were carried out for four values of ρ/m as follows:

	Case 1	*Case 2*	*Case 3*	*Case 4*
ρ/m . .	1/40	1/4	9/8	81/20
α . .	1/5	1/2	3/4	9/10
β . .	−1/4	−1	−3	−9

Case 1 corresponds to a brick wall 9 inches thick and approximately 14 feet high. The other cases are perhaps more likely to occur in submarine blast problems.

The interval $1 \leqslant t \leqslant 2$

It is found, on substituting for $p_0(z)$ and using known relations between α and β, that

$$\rho U(t) = -1+\frac{2\rho}{m(\alpha-\beta)}\left[\frac{1-\alpha}{\alpha}e^{\alpha t}-\frac{1-\beta}{\beta}e^{\beta t}\right]-$$
$$-\frac{16\rho^2}{m^2\pi(\alpha-\beta)}\int\limits_0^{t-1}\left\{\frac{y^{\frac{1}{2}}}{1+y}+\tan^{-1}y^{\frac{1}{2}}\right\}\left\{\left[(1-\alpha)(t-1-y)+\frac{2}{\alpha-\beta}\right]e^{\alpha(t-1-y)}+\right.$$
$$\left.+\left[(1-\beta)(t-1-y)-\frac{2}{\alpha-\beta}\right]e^{\beta(t-1-y)}\right\}dy. \tag{21}$$

The values for $\rho U(t)$ corresponding to case 1 were found to be so small in the first interval that they were not computed in the second interval, as they would not have exhibited the fluctuations due to the additional term sufficiently well. The following is a table of values of $(-1)\rho U(t)$ for the interval $0 \leqslant t \leqslant 2$, and displays the oscillatory nature of the motion. As

there are no spring or damping forces involved, the strip would continue to oscillate slowly back to the rest position.

Table of values of $(-1)\rho U(t)$

t	Case 1	Case 2	Case 3	Case 4	t	Case 1	Case 2	Case 3	Case 4
0	0·000	0·000	0·000	0·000	0·8	0·024	0·203	0·563	0·813
0·1	0·005	0·046	0·192	0·516	0·9	0·024	0·206	0·553	0·795
0·2	0·009	0·086	0·329	0·741	1·0	0·025	0·205	0·538	0·776
0·3	0·013	0·119	0·424	0·820	1·2	..	0·197	0·506	0·772
0·4	0·016	0·146	0·489	0·845	1·4	..	0·192	0·503	0·797
0·5	0·019	0·168	0·530	0·847	1·6	..	0·192	0·520	0·817
0·6	0·021	0·184	0·554	0·840	1·8	..	0·200	0·550	0·836
0·7	0·022	0·196	0·564	0·828	2·0	..	0·216	0·589	0·857

Figs. 1 and 2 are graphs of the pressure on the back of the strip plotted for the interval $0 \leqslant t \leqslant 1$, and contain four sets of curves labelled 1 to 4, corresponding to cases 1 to 4 respectively. Each curve labelled A is for the point $x = 0{\cdot}1$, B for $x = 0{\cdot}3$, and C for $x = 0{\cdot}5$. The discontinuities in slope displayed by each curve represent the arrival of diffracted wave-fronts from one of the edges.

The set of graphs taken together cover the transition from the case of a very heavy (i.e. nearly stationary) strip, for which ρ/m is small, to that of a very light one. As is seen from the graphs, the pressure, $[p_1+p_2-\rho U(t)]$, on the back of the strip is greater for the lighter strips, which means that these would be less effective as shields. However, the magnitude,

$$2[1-p_1-p_2+\rho U(t)],$$

of the pressure discontinuity across the strip is less for the lighter strip, which means that the total force acting on such a strip would be smaller.

Whereas in Fox's problem the pressure on the back of the strip did not become equal to that of the incident pulse, i.e. unity, until time $t = 1$, we see that, in all the present cases, such 'equalization' takes place before this time. (At a time when equalization takes place, the sign of the pressure difference across the strip changes.) The following table shows the approximate time at which equalization first takes place for the various curves:

	Case 1	Case 2	Case 3	Case 4
Curve A . .	0·998	0·95	0·93	0·18
Curve B . .	0·98	0·87	0·75	0·35
Curve C . .	0·98	0·84	0·59	0·51

Sets 3 and 4 exhibit definite maxima, and set 4, in particular, shows some rather striking features. Firstly, equalization occurs very soon after the arrival of the first diffracted wave. Then, after attaining a maximum of between 1·0 and 1·2, the pressure on the back falls off until it is below

unity, rising again above unity soon after the arrival of the second diffracted wave. Thus the strip is being subjected to a force of frequently

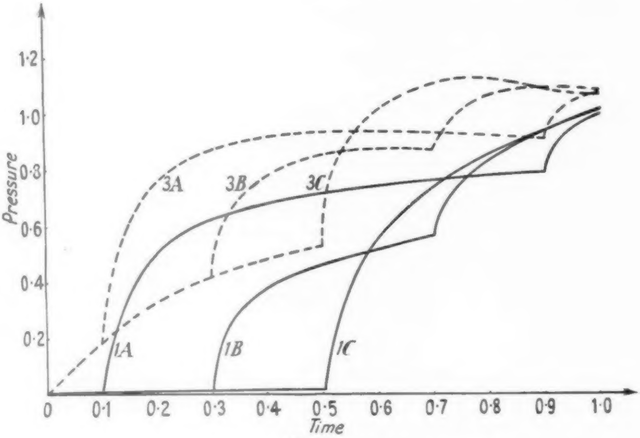

Fig. 1.

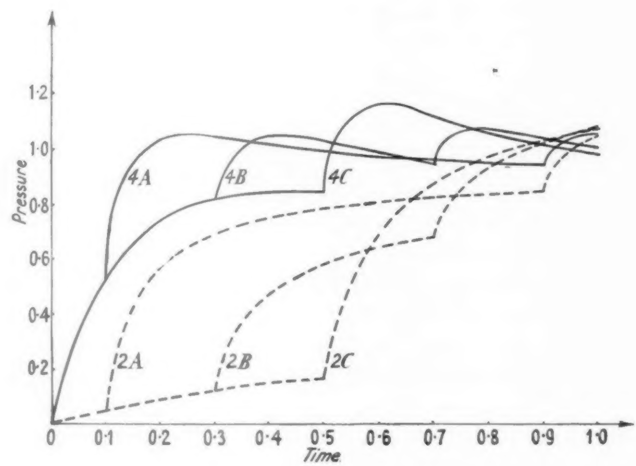

Fig. 2.

varying sign, and it seems likely that this phenomenon would persevere throughout the motion.

Thus, summing up, the heavier strips are subjected to forces of greater magnitude and of mainly constant direction, while the lighter strips are subjected to smaller forces but of more frequently varying direction.

The method which has been used could, of course, be extended to cover a longer period of time, but the additional labour involved would be greater than the result would warrant.

In conclusion, the author wishes to express his gratitude to Dr. F. G. Friedlander and Mr. D. S. Jones for all their help and encouragement in connexion with this paper.

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THE DIFFRACTION OF A SOUND PULSE BY A NON-RIGID SEMI-INFINITE PLANE SCREEN

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SUMMARY

A treatment is given in this paper of the two-dimensional diffraction of a plane-fronted sound pulse, normally incident on a semi-infinite plane screen, each particle of which is able to yield under the action of the incident pressure. Two methods of finding the pressure change across the screen are given, both depending on the method of successive approximations. Graphs are given illustrating the results found by the first method for several types of screen material.

1. Introduction

THE problem considered in this paper is the two-dimensional one of the semi-infinite plane screen subjected to a plane-fronted sound pulse at normal incidence. The solution for the stationary screen is, of course, well known and has been discussed by many authors. The stationary screen is, however, by no means an accurate representation of that employed in practice, as such a screen is neither immobile nor rigid. Although it is not possible to treat the practical problem as it stands, some characteristics can be reproduced by allowing the screen to move in ways which are approximately like those observed.

In a previous paper (**1**) the problem considered was that of diffraction by an infinitely long strip of finite width, capable of motion as a rigid, spring supported body. Such an obstacle, although rigid, still reproduces one feature experienced in practice, in that a disturbance at any point immediately affects the entire screen. In the present paper another aspect of the screen used in practice is considered, namely that of non-rigidity. Now, each particle of the screen is supposed capable of yielding independently under the action of the pulse.

This may be considered as giving some indication of what occurs when a sound pulse strikes an infinitely long strip, clamped along the middle, and able to move according to the laws of flexural vibration. Firstly, neglect of the term due to bending in the equation of motion of the strip will be expected to be a good approximation near the edges of the strip, as there the bending moment will be nearly zero, only becoming large near the middle of the strip. Also, in the interval of time between the incidence of the pulse and the first arrival at one edge of a diffracted wave from the other, the effect can be treated as a superposition of waves arising from

semi-infinite plane screens. Thus, in such an interval the solution obtained in this paper may be expected to be a fairly good guide to the state of affairs near the edge of the strip.

An integral equation is set up for the pressure saltus across the screen. This is then solved by two methods: firstly by a direct application of the method of successive approximations, and secondly by applying the same method to a modified form of the equation. For the purposes of computation the first method is found to give the more satisfactory result, since, although the series so obtained is not so rapidly convergent as that generated by the second method, the terms are more easily calculated, and there is the additional advantage that one set of computations suffices for a fairly wide range of screen materials, whereas the second series has to be recalculated for each separate type.

The first three terms of the series have been computed when the incident pressure pulse is sharp-fronted and of constant unit pressure. Graphs of the pressure saltus are given for four types of screen material showing the modifications introduced by allowing the screen to yield.

2. Formulation and solution of the problem

Let us consider a semi-infinite plane screen represented in the (x, y)-plane when at rest by
$$y = 0 \quad (0 \leqslant x < \infty).$$
Let the total pressure set up when the screen is struck at normal incidence by a pulse $p_0(t+y)$ be $p(x,y,t)$ such that
$$p(x,y,t) = p_0(t+y)+p_1(x,y,t), \tag{1}$$
$p_1(x,y,t)$ being the 'diffracted' pressure. The unit of time has been chosen so that the velocity of sound is unity, and the pressure, p, satisfies the equation
$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2} = \frac{\partial^2 p}{\partial t^2}. \tag{2}$$

Let $v(x,t)$ denote the velocity of the particle of the screen which is a distance x from the edge at time t, displacements being assumed small. Then the boundary condition with which we shall be concerned is
$$m\frac{\partial v}{\partial t} = [p(x,y,t)]_{y=+0}^{y=-0} = p(x,-0,t)-p(x,+0,t), \tag{3}$$
where m is a constant related to the density of the screen.

Also, by equating the acceleration of a particle of the screen and that of the sound wave, we have the relation
$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\left[\frac{\partial p}{\partial y}\right]_{y=0} = -\frac{1}{\rho}q_1(x,t)-\frac{1}{\rho}p_0'(t), \tag{4}$$
where
$$q_1(x,t) = \left[\frac{\partial}{\partial y}p(x,y,t)\right]_{y=0}, \tag{5}$$

and ρ is the density. Using (1) and then substituting in (3) for $\partial v/\partial t$ we obtain

$$\rho[p_1]_{y=-0}^{y=+0} = m[q_1(x,t)+p_0'(t)]. \tag{6}$$

Now according to Friedlander (**2**)

$$p_1(x,y,t) = -\frac{\operatorname{sgn} y}{\pi}\iint \frac{q_1(x',t')}{[(t-t')^2-(x-x')^2-y^2]^{\frac{1}{2}}}\,dx'dt', \tag{7}$$

where the area of integration is defined by

$$t-(x^2+y^2)^{\frac{1}{2}}-x' \leqslant t',$$
$$0 \leqslant t' \leqslant t-[(x-x')^2+y^2]^{\frac{1}{2}}.$$

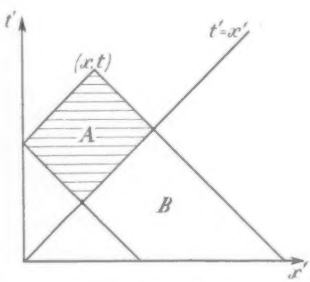


FIG. 1.

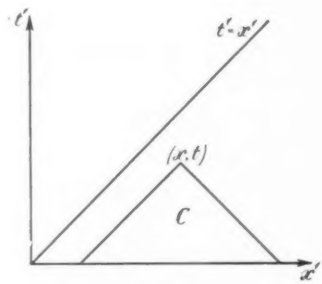


FIG. 2.

Letting $y \to \pm 0$ in this formula, we obtain

$$[p_1]_{y=-0}^{y=+0} = -\frac{2}{\pi}\iint \frac{q_1(x',t')}{[(t-t')^2-(x-x')^2]^{\frac{1}{2}}}\,dx'dt', \tag{8}$$

where the area of integration is now defined by

$$0 \leqslant t-x-x' \leqslant t' \leqslant t-|x-x'|,$$

for $t \geqslant x$, and will be denoted by $A+B$, while for $t \leqslant x$ it is the area $0 \leqslant t' \leqslant t-|x-x'|$, which will be denoted by C (see Figs. 1 and 2).

For the remainder of this section, we shall only be concerned with times $t \geqslant x$, the solution for $t \leqslant x$ being obtained otherwise in section 4.

Eliminating $q_1(x,t)$ from equations (5) and (8), and writing λ for $2\rho/m$, we find that

$$[p_1]_{y=-0}^{y=+0}+\frac{\lambda}{\pi}\iint\limits_{A+B} \frac{[p_1]_{y=-0}^{y=+0}}{\gamma^{\frac{1}{2}}}\,dx'dt' = \frac{2}{\pi}\iint\limits_{A+B} \frac{p_0'(t')}{\gamma^{\frac{1}{2}}}\,dx'dt', \tag{9}$$

where $\gamma = (t-t')^2-(x-x')^2$.

Here it may be noted that $\lambda = 0$ is the rigid screen case, and then (9) is equivalent to the equation set up by Fox (**3**).

This equation can now be treated by the method of successive approximations, i.e. we put

$$[p_1]_{y=-0}^{y=+0} = \phi_0 - \frac{\lambda}{\pi}\phi_1 + \frac{\lambda^2}{\pi^2}\phi_2 + \ldots + \left(-\frac{\lambda}{\pi}\right)^r \phi_r + \ldots, \tag{10}$$

where, for all r, ϕ_r does not depend on λ.

On inserting this in (9) we obtain

$$\phi_0 = \frac{2}{\pi} \iint\limits_{A+B} \frac{p_0'(t')}{\gamma^{\frac{1}{2}}}\, dx'dt', \tag{11}$$

and for $r \geqslant 0$

$$\phi_{r+1} = \iint\limits_{A+B} \frac{\phi_r(x',t')}{\gamma^{\frac{1}{2}}}\, dx'dt'. \tag{12}$$

By performing the necessary integration, we find

$$\phi_0 = 2p_0(t) - \frac{2}{\pi} \int\limits_0^{t-x} \frac{p_0(t')}{t-t'} \left(\frac{x}{t-x-t'}\right)^{\frac{1}{2}} dt'\, H(t-x), \tag{13}$$

where $H(t)$ is the Heaviside unit function defined by

$$\begin{aligned} H(t) &= 1 \quad \text{if } t > 0 \\ &= 0 \quad \text{if } t < 0. \end{aligned} \tag{13a}$$

Thus, when $p_0(t) = H(t)$

$$\phi_0 = 2H(t) - \frac{4}{\pi} \tan^{-1}\left\{\left(\frac{t-x}{x}\right)^{\frac{1}{2}}\right\} H(t-x). \tag{14}$$

This is the term which arises in the theory of diffraction by a stationary strip during the same interval as we are now considering, being, in fact, a combination of solutions for the stationary semi-infinite screen (see Fox's paper (**3**)).

In this case it is found that we can perform one of the integrations involved in finding ϕ_1 and that we can write

$$\begin{aligned} \phi_1 = {} & 2\pi t H(t) - 2\pi(t-x)H(t-x) - \\ & - 4H(t-x)\left\{ t \tan^{-1}\left(\frac{t-x}{x}\right)^{\frac{1}{2}} + \frac{2\sqrt{2}}{\pi}(t-x) \int\limits_0^1 (1-u^2)^{\frac{1}{2}} \tan^{-1}\left(\frac{\sqrt{\epsilon}}{u}\right) du - \right. \\ & \left. - \frac{2}{\pi}(t-x) \int\limits_0^1 (\epsilon+2-u^2)^{\frac{1}{2}} \tan^{-1}\left[\frac{\epsilon^{\frac{1}{2}}}{u}\left(\frac{2-2u^2}{\epsilon+2-u^2}\right)^{\frac{1}{2}}\right] du \right\} \end{aligned} \tag{15}$$

where $\epsilon = 2x/(t-x)$.

When these integrations were carried out (by numerical methods) it was found that

$$\phi_1 = \pi[2tH(t) + (a-2)(t-x)H(t-x)],$$

approximately, where a is roughly $-1{\cdot}6$ and varies slightly with x. This was used to obtain an approximate value Φ_2 for ϕ_2. Thus we find that

$$\Phi_2 = 2\pi t^2\left[\frac{\pi}{2}H(t)-\tan^{-1}\left\{\left(\frac{t-x}{x}\right)^{\frac{1}{2}}\right\}H(t-x)\right]+$$
$$+\tfrac{1}{3}\pi[2(5t-2x)x^{\frac{1}{2}}(t-x)^{\frac{1}{2}}+4(a-2)(t-x)(2x)^{\frac{1}{2}}(t-x)^{\frac{1}{2}}]H(t-x). \quad (16)$$

It can be shown that the series is convergent, but the proof is omitted.

If, for the moment, we consider the strip problem, recalling that the solution just obtained will be a good approximation near the edges during the interval in which the effect can be considered as a superposition of two semi-infinite screen solutions, we may thereby obtain some idea of the error involved as follows.

In the (x,y)-plane let the strip extend from $x = 0$ to $x = 2$.

The equation of motion to replace equation (3) would be

$$m\frac{\partial v}{\partial t}+B\frac{\partial^4}{\partial x^4}\int\limits_0^t v(x,t')\,dt' = [p(x,y,t)]_{y=+0}^{y=-0}, \quad (17)$$

where m and B are constants, and, of course, further restrictions on v and its derivatives are imposed by the support.

From (8), (10), (11), and (12) it can be seen that the expansion (10) can be rephrased in terms of q_1, and is then equivalent to setting

$$q_1(x,t) = -p_0'(t)+O\left(\frac{\lambda}{\pi}\right).$$

Using this as a basis for estimating the bending term on the left-hand side of (17) (in a purely formal manner), we see that it gives rise to a term of order $O(\lambda B/\rho\pi)$, which is a second-order effect. Thus, to the first order at least, our approximation should be valid.

3. Calculations

The quantities ϕ_0, ϕ_1, and Φ_2 as defined by equations (14), (15), and (16) were computed for values of x from $0{\cdot}2$ to $1{\cdot}0$ at intervals of $0{\cdot}2$, and of t from 0 to $2{\cdot}0$ at similar intervals.

Taking this scale for x and t necessitates care in fixing the value of λ to make the computation conform to a physical system. The actual value required is

$$\lambda = 2l\times(\text{density of air})\times(\text{density of screen material})^{-1}\times$$
$$\times(\text{thickness of screen})^{-1},$$

where l is the equivalent of the unit of length in centimetres (e.g. if we were dealing with the strip, $2l$ would be the actual width of it).

Hence the expression

$$\phi_0 - \frac{\lambda\phi_1}{\pi} + \frac{\lambda^2\Phi_2}{\pi^2}$$

was calculated for $\lambda = 1/40$, $1/20$, $1/10$, and $1/4$. It was found that for $\lambda = 1/40$, the third term was making a maximum contribution of 0·001, while for $\lambda = 1/4$ it was making a contribution of 0·100.

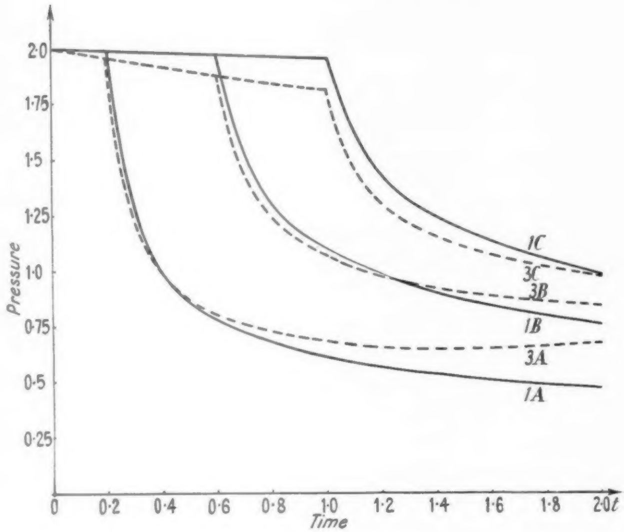
Figs. 3 and 4 are graphs illustrating the results obtained, and contain four sets of curves, labelled 1 to 4 to correspond to the cases $\lambda = 1/40$, $1/20$, $1/10$, and $1/4$, respectively. Each curve labelled A is for the point $x = 0{\cdot}2$, B for $x = 0{\cdot}6$, and C for $x = 1{\cdot}0$, curves for intermediate points being omitted for the sake of clarity.

The effect of the motion of the screen is already quite marked in the set of curves 2. Curves 1 show very little deviation from the corresponding curves for the stationary screen, as they represent a fairly weighty wall—the value of λ taken being as for a 9-in. thick brick wall about 15 ft. high.

A qualitative comparison with the corresponding results in the previous paper (**1**) shows that the pressure discontinuity here is greater than that previously calculated, except during the period of time which elapses between the arrival of the pulse and an instant very soon after the arrival of the diffracted wave, when the reverse is true. This means that, after this initial interval, the new type of screen is more effective as a shield although the force on it is correspondingly larger.

A similar comparison amongst the sets of curves given here reveals a more complicated state of affairs. The set of graphs taken together represents a survey of the transition from the case of a very heavy screen to a much lighter one. Now, near the time $t = 2$, the pressure discontinuity is larger for the lighter screens, except near the point $x = 1{\cdot}0$ (curves C)—which, for practical purposes, is the least important, being too far from the edge of the screen to be a good representation of the corresponding point on the strip (the mid-point). However, near the beginning of the motion, and for some time after the arrival of the diffracted wave, the pressure discontinuity is smaller for the lighter screens. The instant at which one condition gives way to the other varies appreciably as we make the transition from the heavier to the lighter screen. Thus we can only say that the heavier screen would be the more effective as a shield in the earlier part of the motion, while later the lighter screen would be the more effective.

No computation has been attempted for points off the screen, but it is doubtful whether any new results of appreciable physical interest would be yielded, as it is at points on the screen that the strongest effects are to be expected.


2·0
1·75
1·5
1·25
1·0
0·75
0·5
0·25
Pressure
0
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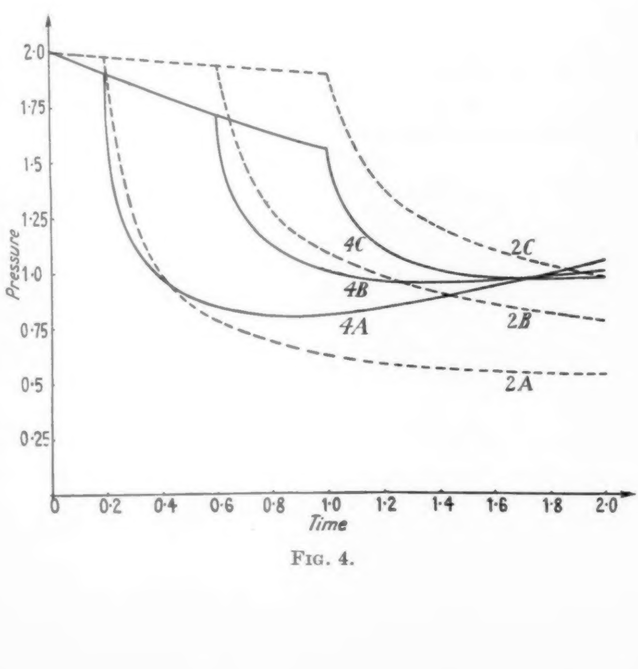
0·4
0·6
0·8
1·0
1·2
1·4
1·6
1·8
2·0 t
Time
1C
3C
3B
1B
3A
1A

Fig. 3.


2·0
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1·0
0·75
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Pressure
0
0·2
0·4
0·6
0·8
1·0
1·2
1·4
1·6
1·8
2·0
Time
4C
2C
4B
4A
2B
2A

Fig. 4.

4. An alternative approach

An alternative method of dealing with equation (9) suggests itself when this equation is written out for $t \leqslant x$. It is then

$$[p_1]_{y=-0}^{y=+0}+\frac{\lambda}{\pi}\iint\limits_C \frac{[p_1]_{y=-0}^{y=+0}}{\gamma^{\frac{1}{2}}}\,dx'dt' = \frac{2}{\pi}\iint\limits_C \frac{p_0'(t')}{\gamma^{\frac{1}{2}}}\,dx'dt'. \tag{18}$$

We note that the area of integration now involved is C, and this is precisely the area of integration involved, whatever the time t, for the case of normal reflection by a yielding infinite plane screen.

The physical interpretation of this is that the effect which diffraction makes upon the screen does not travel along the screen quicker than through the air. In other words, a point on the screen does not 'know' of the existence of the edge until disturbances set up there have had time to propagate to the point in question through the surrounding air.

In consequence of these remarks, let us consider for a moment the problem of a plane screen of infinite extent, represented by the plane $y = 0$, at rest, when the incident pressure field is $P_0(t+y)$ giving rise to a total pressure $P(x,y,t)$ such that

$$P(x,y,t) = P_0(t+y)+P_1(x,y,t), \tag{19}$$

where P_1 is the 'reflected' pressure.

Suppose that we had a boundary condition

$$\left[\frac{\partial P}{\partial y}\right]_{y=0} = \lambda[P]_{y=+0}$$

i.e.

$$\left[\frac{\partial P_1}{\partial y}\right]_{y=0} = \lambda[P_1]_{y=+0}+\lambda P_0(t)-P_0'(t). \tag{20}$$

Again using Friedlander's result,

$$[P_1]_{y=-0}^{y=+0} = 2[P_1]_{y=+0} = -\frac{2}{\pi}\iint\limits_C \frac{[\partial P_1/\partial y]_{y=0}}{\gamma^{\frac{1}{2}}}\,dx'dt'. \tag{21}$$

Eliminating $[\partial P_1/\partial y]_{y=0}$ from (21) and (20) we have

$$[P_1]_{y=-0}^{y=+0}+\frac{\lambda}{\pi}\iint\limits_C \frac{[P_1]_{y=-0}^{y=+0}}{\gamma^{\frac{1}{2}}}\,dx'dt' = -\frac{2}{\pi}\iint\limits_C \frac{[\lambda P_0(t')-P_0'(t')]}{\gamma^{\frac{1}{2}}}\,dx'dt', \tag{22}$$

and this is precisely the same equation as (18) if we put

$$\lambda P_0(t)-P_0'(t) = -p_0'(t). \tag{23}$$

Now, since this problem is one of pure reflection, we can assume that P will be of the form $F(t-y)$ for $y \geqslant 0$, and then applying (20) and (23) we find that

$$F(t) = p_0(t)-\lambda\int\limits_0^t p_0(t-z)e^{-\lambda z}\,dz. \tag{24}$$

Now returning to equation (9) and using the value just obtained for $[p_1]_{y=-0}^{y=+0}$ when $t \leqslant x$, we may write

$$[p_1]_{y=-0}^{y=+0}+\frac{\lambda}{\pi}\iint\limits_A \frac{[\phi_1]_{y=-0}^{y=+0}}{\gamma^{\frac{1}{2}}}\,dx'dt' = \frac{2}{\pi}\iint\limits_A \frac{p_0'(t')}{\gamma^{\frac{1}{2}}}\,dx'dt' - \frac{2}{\pi}\iint\limits_B \frac{F'(t')}{\gamma^{\frac{1}{2}}}\,dx'dt',$$

and hence, carrying out the integral over B,

$$[p_1]_{y=-0}^{y=+0}+\frac{\lambda}{\pi}\iint\limits_A \frac{[p_1]_{y=-0}^{y=+0}}{\gamma^{\frac{1}{2}}}\,dx'dt' = \frac{2}{\pi}\iint\limits_A \frac{p_0'(t')}{\gamma^{\frac{1}{2}}}\,dx'dt' +$$

$$+\frac{2}{\pi}\int\limits_{(t-x)/2}^{(t+x)/2} \frac{F(t')}{t-t'}\sqrt{\left(\frac{t-x}{t+x-2t'}\right)}\,dt' - \frac{2}{\pi}\int\limits_0^{(t-x)/2} \frac{F(t')}{t-t'}\sqrt{\left(\frac{x}{t-x-t'}\right)}\,dt'. \quad (25)$$

The double integral on the right-hand side has not been evaluated for a reason which will become apparent later on when a special value is taken for $p_0(t)$.

We now make a change of variable by putting

$$\left.\begin{aligned} t+x &= 2\alpha \\ t-x &= 2\beta \end{aligned}\right\} \text{ throughout and } \left.\begin{aligned} t'+x' &= 2\alpha' \\ t'-x' &= 2\beta' \end{aligned}\right\} \text{ in the double integrals.}$$

We also write $\quad \Psi(\alpha,\beta)$ for $[p_1\{x(\alpha,\beta),t(\alpha,\beta)\}]_{y=-0}^{y=+0}$. (26)

Then we have

$$\Psi(\alpha,\beta)+\frac{\lambda}{\pi}\int\limits_0^{\beta}\int\limits_{\beta}^{\alpha} \frac{\Psi(\alpha',\beta')}{[(\alpha-\alpha')(\beta-\beta')]^{\frac{1}{2}}}\,d\alpha'd\beta' = \psi_0(\alpha,\beta), \quad (27)$$

where

$$\psi_0(\alpha,\beta) = \frac{2}{\pi}\int\limits_0^{\beta}\int\limits_{\beta}^{\alpha} \frac{p_0'(\alpha'+\beta')}{[(\alpha-\alpha')(\beta-\beta')]^{\frac{1}{2}}}\,d\alpha'd\beta' +$$

$$+\frac{2}{\pi}\int\limits_{\beta}^{\alpha} \frac{F(t)}{\alpha+\beta-t}\left(\frac{\beta}{\alpha-t}\right)^{\frac{1}{2}} dt - \frac{2}{\pi}\int\limits_0^{\beta} \frac{F(t)}{\alpha+\beta-t}\left(\frac{\alpha-\beta}{2\beta-t}\right)^{\frac{1}{2}} dt. \quad (28)$$

For the sake of simplicity, let us consider the special case of $p_0(t) = H(t)$, this being the one usually used for computation. Then the double integral over the area A on the right-hand side of equation (28) vanishes since $p_0'(t') = \delta(t') = 0$ unless $t' = 0$, and this lies outside the area A; and ψ_0 becomes

$$\psi_0(\alpha,\beta) = \frac{2}{\pi}\int\limits_{\beta}^{\alpha} \frac{e^{-\lambda t}}{\alpha+\beta-t}\left(\frac{\beta}{\alpha-t}\right)^{\frac{1}{2}} dt - \frac{2}{\pi}\int\limits_0^{\beta} \frac{e^{-\lambda t}}{\alpha+\beta-t}\left(\frac{\alpha-\beta}{2\beta-t}\right)^{\frac{1}{2}} dt. \quad (29)$$

We now put

$$\Psi(\alpha,\beta) = \psi_0(\alpha,\beta) - \frac{\lambda\psi_1(\alpha,\beta)}{\pi} + \dots + \frac{(-\lambda)^r\psi_r(\alpha,\beta)}{\pi^r} + \dots, \tag{30}$$

where, unlike the ϕ_r, the ψ_r are now functions of λ as well. On substituting in (27) we obtain

$$\psi_{r+1}(\alpha,\beta) = \int\limits_0^\beta \int\limits_\beta^\alpha \frac{\psi_r(\alpha',\beta')}{[(\alpha-\alpha')(\beta-\beta')]^{\frac{1}{2}}}\, d\alpha' d\beta', \tag{31}$$

for $r \geqslant 0$. Of course, the same expansion still applies when ψ_0 is given by (28), but then the actual integrals which appear always involve one more integration than in those appearing when ψ_0 is given by (29).

The series (30) can be seen to be rapidly convergent as follows. It can be shown by iterating equation (27) that

$$\psi_{r+2}(\alpha,\beta) = \int\limits_0^\beta \int\limits_{\beta'}^\alpha \psi_r(\alpha',\beta') k(\alpha,\beta,\alpha',\beta')\, d\alpha' d\beta', \tag{32}$$

where

$$\left.\begin{aligned} k(\alpha,\beta,\alpha',\beta') &= 4\sin^{-1}\left(\frac{\alpha-\beta}{\alpha-\alpha'}\right)^{\frac{1}{2}} \sin^{-1}\left(\frac{\alpha'-\beta'}{\beta-\beta'}\right)^{\frac{1}{2}} \quad \text{for } \beta' \leqslant \alpha' \leqslant \beta \\ &= \pi^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \beta \leqslant \alpha' \leqslant \alpha \end{aligned}\right\}. \tag{33}$$

Then, if we assume that $|\psi_0| \leqslant M$ (some constant) for some range of α and β, application of (33) shows that

$$\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{\pi^{2n}} |\psi_{2n}| \leqslant M\left[I_0\{2\lambda\surd(\alpha\beta)\} - \frac{\beta}{\alpha} I_2\{2\lambda\surd(\alpha\beta)\}\right] \tag{34}$$

where I_0, I_2 are Bessel functions.

Also, taking $|\psi_1| \leqslant 4M\alpha$ [it is actually $\leqslant 4M\beta^{\frac{1}{2}}(\alpha-\beta)^{\frac{1}{2}}$] it is found that

$$\sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{\pi^{2n+1}} |\psi_{2n+1}| \leqslant \frac{4M}{\pi}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} \left[I_1\{2\lambda\surd(\alpha\beta)\} - \frac{\beta^2}{\alpha^2} I_3\{2\lambda\surd(\alpha\beta)\}\right]. \tag{35}$$

The factor $e^{-\lambda t}$ in equation (29) makes the series all the more rapidly convergent. Unfortunately, this is also the chief disadvantage of the method, as, for purposes of computation, the integrals in (29), for example, would have to be recalculated for each separate value of λ, whereas, in the method adopted in section 2, one computation is sufficient for a wide range of values of λ. For this reason, no computation has been attempted on this series.

In conclusion, the author wishes to thank Dr. F. G. Friedlander and Mr. D. S. Jones for all their assistance in connexion with this paper.

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BOUNDARY-VALUE PROBLEMS OF THE ELASTIC HALF-PLANE

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SUMMARY

In this investigation we employ a combination of complex potential and Fourier integral methods to solve problems of generalized plane stress of semi-infinite plates with one straight boundary. In the first part of this paper relations between the complex potentials $\Omega(z)$, $\omega(z)$ are obtained suitable for giving any desired conditions of stress or displacement along the real axis, thus reducing problems of the half-plane $y \geqslant 0$ to the determination of the single potential function $\Omega(z)$. A simple method is given of obtaining $\Omega(z)$ from specified conditions along the real axis and at infinity. The method is applied to the solution of problems of the half-plane $y \geqslant 0$ in equilibrium with specified stresses or displacements along the straight boundary and balancing stresses at infinity and to the problem of the half-plane subjected to an interior force and with the straight boundary free from displacement.

Introduction

For the majority of two-dimensional elastic problems, solutions in terms of complex potentials are simpler and more informative than Airy stress-function solutions. However, a difficulty in the use of complex potentials lies in the lack of general methods of obtaining them. For the Airy stress-function treatment the results of years of research into the solution of differential equations is available. Whilst the tentative method has proved powerful (see Stevenson (**1**), (**2**)) it is important to establish general methods of solution in terms of complex potentials. For problems involving material bounded by a simple closed contour, an elegant method for the determination of the potentials in terms of given boundary stresses or displacements has been given by Muschelisvili (**3**, **4**), who has also given a treatment of the elastic half-plane. The publications (**5**, **6**) containing the latter investigations are not easily obtained. However, Muschelisvili's basic assumption that the straight boundary is stress-free except for a finite number of finite segments shows that the treatment is not exhaustive of all possibilities. Sneddon (**7**, **8**) and Girkmann (**9**) have considered the problem of the half-plane with specified stresses over the straight boundary in terms of the Airy stress function. The treatments are purely formal and introduce unnecessary analytical difficulties.

Fundamental equations

The following investigations are confined to elastic material in equilibrium, subjected neither to body forces nor body couples, and in a state of

generalized plane stress. The stress and displacement components may be expressed in terms of complex potentials $\Omega(z)$, $\omega(z)$ as follows (Stevenson (**1**, **2**))

$$2\Theta = 2(\widetilde{xx}+\widetilde{yy}) = \Omega'(z)+\bar{\Omega}'(\bar{z}), \tag{1}$$

$$-2\bar{\Phi} = -2(\widetilde{xx}-\widetilde{yy}-2i\widetilde{xy}) = \bar{z}\Omega''(z)+\omega''(z), \tag{2}$$

$$8\mu D = 8\mu(u+iv) = \kappa\Omega(z)-z\bar{\Omega}'(\bar{z})-\bar{\omega}'(\bar{z}), \tag{3}$$

where κ is an elastic constant and $\Omega(z)$, $\omega(z)$ have continuous derivatives of the second order at all points in the interior of the material. From (1), (2) we see that

$$2(\Theta-\bar{\Phi}) = 4(\widetilde{yy}+i\widetilde{xy}) = \Omega'(z)+\bar{\Omega}'(\bar{z})+\bar{z}\Omega''(z)+\omega''(z). \tag{4}$$

Notation. Throughout this work considerable advantage is gained by choosing the straight boundary of the material as the real axis and we shall write

$$\widetilde{pq}_{y=0} = \widetilde{pq}^{0} \text{ etc.} \tag{5}$$

Relations between the complex potentials suitable for giving any desired conditions of stress or displacement along the real axis

(i) Let

$$\omega(z) = -z\Omega(z)+2\int\Omega(z)\,dz. \tag{6}$$

Equation (4) becomes

$$4(\widetilde{yy}+i\widetilde{xy}) = \Omega'(z)+\bar{\Omega}'(\bar{z})-2iy\Omega''(z). \tag{7}$$

Thus

$$2\widetilde{yy}^{0} = \mathrm{re}[\Omega'(z)]_{y=0} = \mathrm{re}[\Omega'(x)], \qquad \widetilde{xy}^{0} = 0. \tag{8}$$

We asume, in general, that $\lim y\Omega''(z) = 0$ at all points of the real axis. We may relax this condition to include those cases in which this limit exists but is not zero or unique at a finite number of points of the real axis. From (8) we see that this combination gives zero shear over the real axis and if we require

$$\widetilde{yy}^{0} = f(x) \tag{9}$$

we must choose $\Omega(z)$ so that

$$\mathrm{re}[\Omega'(x)] = 2f(x). \tag{10}$$

(ii) Let

$$\omega(z) = -z\Omega(z). \tag{11}$$

Equation (4) becomes

$$4(\widetilde{yy}+i\widetilde{xy}) = \bar{\Omega}'(\bar{z})-\Omega'(z)-2iy\Omega''(z). \tag{12}$$

Thus

$$\widetilde{yy}^{0} = 0, \qquad 2\widetilde{xy}^{0} = -\mathrm{im}[\Omega'(x)]. \tag{13}$$

If we require

$$\widetilde{xy}^{0} = \phi(x) \tag{14}$$

we must choose $\Omega(z)$ so that

$$-\mathrm{im}[\Omega'(x)] = 2\phi(x). \tag{15}$$

(iii) Let

$$\omega(z) = -z\Omega(z)+(1-\kappa)\int\Omega(z)\,dz. \tag{16}$$

Equation (3) becomes

$$8\mu D = \kappa[\Omega(z)+\bar{\Omega}(\bar{z})]-2iy\bar{\Omega}'(\bar{z}). \tag{17}$$

Thus $$4\mu u^0 = \kappa\,\mathrm{re}[\Omega(x)], \qquad v^0 = 0. \tag{18}$$

We have assumed that $\lim y\Omega'(z) = 0$ at all points of the real axis but this may be relaxed as above.

If we require $$u^0 = \psi(x), \tag{19}$$

we must choose $\Omega(z)$ so that

$$\mathrm{re}[\Omega(x)] = \frac{4\mu}{\kappa}\psi(x). \tag{20}$$

(iv) Let $$\omega(z) = -z\Omega(z)+(1+\kappa)\int\Omega(z)\,dz. \tag{21}$$

Equation (3) becomes

$$8\mu D = \kappa[\Omega(z)-\bar{\Omega}(\bar{z})]-2iy\bar{\Omega}'(\bar{z}). \tag{22}$$

Thus $$u^0 = 0, \qquad 4\mu v^0 = \kappa\,\mathrm{im}[\Omega(x)]. \tag{23}$$

If we require $$v^0 = \chi(x) \tag{24}$$

we must choose $\Omega(z)$ so that

$$\mathrm{im}[\Omega(x)] = \frac{4\mu}{\kappa}\chi(x). \tag{25}$$

Equations (10), (15), (20), (25) reduce problems of the half-plane $y \geqslant 0$ with specified stresses or displacements along $y = 0$ to the determination of functions which are analytic in the half-plane $y > 0$, of suitable orders of magnitude at infinity, and which have specified real or imaginary parts on the real axis. We will consider, in general, problems in which the potentials at infinity are to be of orders such that

$$\Omega'(z) = O(z^{-1}), \qquad \omega''(z) = O(z^{-1}). \tag{26}$$

Thus the stresses at infinity are $O(z^{-1})$. These are the lowest possible orders if the stresses applied along the real axis have a non-zero resultant.

Determination of the function $F(z)$ which is analytic in the region $y > 0$ and has a specified real or imaginary part along the real axis

Clearly we need consider only the case for which the real part is given on the real axis.

Let $f(x)$ satisfy conditions sufficient for it to be expressible as a Fourier integral (**10**), e.g. let $f(x)$ be of bounded variation and at each point

$$f(x) = \tfrac{1}{2}[f(x-0)+f(x+0)] \tag{27}$$

and $$\int\limits_{-\infty}^{\infty} |f(x)|\,dx < \infty. \tag{28}$$

Consider the function

$$F(z) = G(x,y)+iH(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f_T(u)e^{izu}\,du, \tag{29}$$

where

$$f_T(u) = \int_{-\infty}^{\infty} f(t)e^{-iut}\,dt. \tag{30}$$

It can easily be shown that, subject to (27), (28), $F(z)$ has the following properties:

$$(a)\quad G(x,0) = f(x) \quad (\text{see } (\mathbf{5})). \tag{31}$$

$$(b)\quad \lim_{y\to 0+} G(x,y) = f(x). \tag{32}$$

$$(c)\quad F(z) \text{ is analytic in } y>0. \tag{33}$$

$$(d)\quad F(z) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{f(t)\,dt}{z-t} \text{ in } y>0. \tag{34}$$

If, in addition to (28), we assume that

$$\int_{-\infty}^{\infty} |xf(x)|\,dx < \infty. \tag{35}$$

$$(e)\quad F(z) = O(z^{-1}) \text{ at infinity in } y>0. \tag{36}$$

Thus, in general, $F(z)$ is the required function which is analytic in the upper half-plane and has real part $f(x)$ on the real axis. Thus equations (10), (15), (20), (25) are satisfied respectively by choosing

$$\Omega'(z) = I(z) = \frac{2}{\pi}\int_{0}^{\infty} f_T(u)e^{izu}\,du, \tag{37}$$

$$\Omega'(z) = J(z) = -\frac{2i}{\pi}\int_{0}^{\infty} \phi_T(u)e^{izu}\,du, \tag{38}$$

$$\Omega(z) = \frac{4\mu}{\kappa\pi}\int_{0}^{\infty} \psi_T(u)e^{izu}\,du, \tag{39}$$

$$\Omega(z) = \frac{4\mu i}{\kappa\pi}\int_{0}^{\infty} \chi_T(u)e^{izu}\,du. \tag{40}$$

Applications of the above theory

The following examples illustrate the simplicity of the above theory.

(i) Given $$\widetilde{yy}^0 = f(x) = \begin{cases} \dfrac{|x|}{b}-1, & |x|<b, \\ 0, & |x|>b, \end{cases} \qquad \widetilde{xy}^0 = 0. \tag{41}$$

From (37)

$$I(z) = \frac{2i}{\pi b}[2z\log z-(z-b)\log(z-b)-(z+b)\log(z+b)]. \tag{42}$$

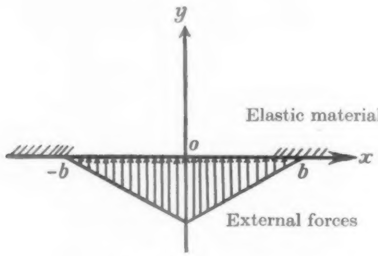

FIG. 1.

The integration may be performed most simply by means of (34), whence

$$\pi bi\Omega(z) = (z-b)^2\log(z-b)+(z+b)^2\log(z+b)-2z^2\log z. \tag{43}$$

From (6)

$$3\pi bi\omega(z) = 8b^2z+2z^3\log z-(z+2b)(z-b)^2\log(z-b)-$$
$$-(z-2b)(z+b)^2\log(z+b). \tag{44}$$

At infinity $\Omega(z) = O(\log z)$, $\omega(z) = O(z\log z)$, so that (26) is satisfied. The functions $\Omega(z)$, $\Omega'(z)$, $\Omega''(z)$, $\omega(z)$, $\omega'(z)$ are continuous in $y \geqslant 0$ so that the stresses given by (1), (2) are finite and uniform in $y \geqslant 0$ and the displacements given by (3) are finite and uniform in the finite part of the upper half-plane but are necessarily logarithmically infinite at infinity.

(ii) Given

$$\widetilde{xy}^0 = \phi(x) = \begin{cases} 2\left(\dfrac{x}{b}-1\right), & \tfrac{1}{2}b < x < b \\ -2\dfrac{x}{b}, & -\tfrac{1}{2}b < x < \tfrac{1}{2}b, \qquad \widetilde{yy}^0 = 0 \\ 2\left(\dfrac{x}{b}+1\right), & -b < x < -\tfrac{1}{2}b \\ 0, & |x| > b. \end{cases} \tag{45}$$

From (38), (34)

$$J(z) = \frac{4}{\pi b}[(z+b)\log(z+b)-(z-b)\log(z-b)+$$
$$+(2z-b)\log(z-\tfrac{1}{2}b)-(2z+b)\log(z+\tfrac{1}{2}b)]. \tag{46}$$

The potentials $\Omega(z)$, $\omega(z)$ may now be evaluated from (38), (11). In this case $J(z) = O(z^{-2})$ so that $\Omega(z) = O(z^{-1})$, $\omega(z) = O(1)$ at infinity. Since the

stresses applied over $y = 0$ are in equilibrium, it might be expected that the stresses at infinity would be of a lower order. Uniqueness of the solution (**11**) shows that this cannot be the case.

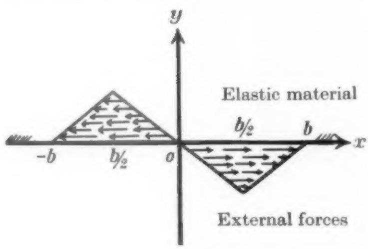

FIG. 2.

(iii) Given $$u^0 = \psi(x) = \frac{1}{(x-b)^2+a^2}, \qquad v^0 = 0, \tag{47}$$

where a and b are real and positive. From (39)

$$\Omega(z) = \frac{4\mu i}{a\kappa(z-b+ia)}. \tag{48}$$

The second potential $\omega(z)$ may now be obtained from (16).

(iv) Given $$u^0 = \psi(x) = \frac{x-b}{(x-b)^2+a^2}, \qquad v^0 = 0. \tag{49}$$

In this case $\psi(x)$ does not satisfy (28) but may still be expressed as a Fourier integral (see (**10**), p. 16). From (39)

$$\Omega(z) = \frac{4\mu}{\kappa(z-b+ia)}, \tag{50}$$

and $\omega(z)$ may be obtained from (16).

(v) Given $$u^0 = \psi(x) = \log[(x-b)^2+a^2], \qquad v^0 = 0. \tag{51}$$

The function $\psi(x)$ cannot be expressed as a Fourier integral. We assume tentatively that

$$\Omega(z) = \frac{8\mu}{\kappa}\log[z-b+ia] \tag{52}$$

and $\omega(z)$ is as given in (16). We see immediately that (20) is satisfied, so that (52) is the required potential.

(vi) Given $$v^0 = \chi(x) = \frac{1}{(x-b)^2+a^2}, \qquad u^0 = 0. \tag{53}$$

From (40) $$\Omega(z) = -\frac{4\mu}{\kappa a[z-b+ia]} \tag{54}$$

and $\omega(z)$ is given by (21).

(vii) Given $$v^0 = \chi(x) = \frac{x-b}{[(x-b)^2+a^2]}, \qquad u^0 = 0. \tag{55}$$

From (40) $$\Omega(z) = \frac{4\mu i}{\kappa[z-b+ia]} \tag{56}$$

and $\omega(z)$ is given by (21).

(viii) Given

$$v^0 = \chi(x) = \log[(x-b)^2+a^2], \qquad u^0 = 0. \tag{57}$$

We again revert to a semi-tentative method. Let

$$\Omega(z) = \frac{8\mu i}{\kappa}\log[z-b+ia]. \tag{58}$$

If $\omega(z)$ is given by (21) we see that (25) is satisfied so that (58) gives the required potential.

Solutions (iii)–(viii) will be applied to the solution of the problem of the half-plane subjected to an isolated interior force and with the straight boundary free from displacement.

The potentials giving an isolated force $F = X+iY$ at $z = b+ia$ in an infinite plate are (see (**1**))

$$\Omega(z) = -\nu F\log[z-b-ia], \tag{59}$$

$$\omega(z) = \nu\kappa\bar{F}(z-b-ia)\log(z-b-ia)+\nu F(b-ia)\log(z-b-ia), \tag{60}$$

where $$\nu = \frac{2}{\pi(\kappa+1)}. \tag{61}$$

Using equation (3) and setting $y = 0$ we obtain

$$8\mu u^0 = -\frac{2\nu a^2X}{(x-b)^2+a^2}-\frac{2\nu aY(x-b)}{(x-b)^2+a^2}-\nu\kappa X\log[(x-b)^2+a^2]+\nu X(1-\kappa), \tag{62}$$

$$8\mu v^0 = \frac{2\nu a^2Y}{(x-b)^2+a^2}-\frac{2\nu aX(x-b)}{(x-b)^2+a^2}-\nu\kappa Y\log[(x-b)^2+a^2]-\nu Y(1-\kappa). \tag{63}$$

We require to annul these displacements by the introduction of potentials which have no singularities in the upper half-plane. From the results given in (iii)–(viii) and obvious rigid body displacement terms we obtain for the required additional potentials

$$\Omega(z) = \nu F\log(z-b+ia)+\frac{2\nu ia\bar{F}}{\kappa(z-b+ia)}, \tag{64}$$

$$\omega(z) = -z\Omega(z)+\left[\frac{2\nu ia\bar{F}}{\kappa}+\nu(F-\kappa\bar{F})(z-b+ia)\right]\log(z-b+ia). \tag{65}$$

It is clear from the above solution that we may solve problems which involve terms in the boundary stresses or displacements which are not expressible as Fourier integrals. We have only to introduce potentials

which give these terms at infinity. The latter potentials may give any additional terms which are expressible as Fourier integrals so that the choice of suitable potentials should not be difficult. The remaining terms may be dealt with by the general method given above.

The above treatment of the elastic half-plane has the additional advantage that it leads on to a comprehensive treatment of the infinite elastic strip in terms of complex potentials.

A brief treatment in terms of complex potentials of the half-plane with specified stresses over the straight boundary has been given by Milne-Thomson (**12**). In terms of the notation used above Milne-Thomson's statements imply that for any given function $\widetilde{yy}^0 = f(x)$, the corresponding function $I(z)$ is obvious. In the only application given he chooses $f(x)$ to be constant so that the need for determining $I(z)$ does not arise. The question of uniqueness of $I(z)$ is not considered. Further, he introduces a single function of z to take care of conditions at infinity without showing that this is, in general, sufficient to give specified stresses at infinity.

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SOLUTION OF TWO-DIMENSIONAL ELASTIC PROBLEMS BY CONFORMAL MAPPING ON TO A HALF-PLANE

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SUMMARY

In this paper we give a general method of solution of problems of generalized plane stress of plate-like material bounded by a single unclosed contour. The theory is restricted to material of such a form that conformal mapping on to a half-plane is possible. That part of the boundary which is mapped on to the straight boundary of the half-plane is subjected to specified stresses or displacements, equilibrium being maintained by evanescent stresses at infinity. The method is illustrated by consideration of material with a parabolic boundary subjected to simple boundary loading.

Fundamental equations

THE notation is that of Stevenson (**1**, **2**). We use complex variables $z = x+iy$, $\zeta = \xi+i\eta$ related by the equation $z = z(\zeta)$, where $z'(\zeta) \neq 0$ in

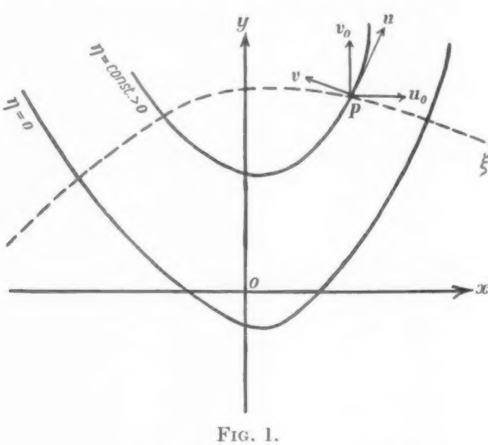


FIG. 1.

$\eta \geqslant 0$; accents denote derivatives. Elastic material is chosen to occupy the region $\eta \geqslant 0$ of the z-plane and the theory follows closely that given by the present writer for treatment of the corresponding problems of the elastic half-plane. In terms of complex potentials $\Omega(\zeta)$, $\omega(\zeta)$ the stresses are given by the equations

$$2\Theta = 2(\breve{\xi\xi}+\breve{\eta\eta}) = \frac{\Omega'(\zeta)}{z'(\zeta)}+\frac{\bar{\Omega}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})} \tag{1}$$

$$-2\bar{\Phi} = -2(\breve{\xi\xi}-\breve{\eta\eta}-2i\breve{\xi\eta}) = \left[\bar{z}(\bar{\zeta})\frac{d}{d\zeta}\frac{\Omega'(\zeta)}{z'(\zeta)}+\frac{d}{d\zeta}\frac{\omega'(\zeta)}{z'(\zeta)}\right]\Big/\bar{z}'(\bar{\zeta}), \tag{2}$$

so that

$$2(\Theta-\bar{\Phi}) = 4(\breve{\eta\eta}+i\breve{\xi\eta}) = \frac{\Omega'(\zeta)}{z'(\zeta)}+\frac{\bar{\Omega}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}+\left[\bar{z}(\bar{\zeta})\frac{d}{d\zeta}\frac{\Omega'(\zeta)}{z'(\zeta)}+\frac{d}{d\zeta}\frac{\omega'(\zeta)}{z'(\zeta)}\right]\Big/\bar{z}'(\bar{\zeta}). \tag{3}$$

Let the displacement $D_0 = u_0+iv_0$ referred to rectangular Cartesian axes $O(x,y)$ and have components u, v in the directions of ξ and η increasing respectively. We write $D = u+iv$. Then (**1**), (**2**)

$$8\mu D_0 = 8\mu(u_0+iv_0) = \kappa\Omega(\zeta)-z(\zeta)\frac{\bar{\Omega}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}-\frac{\bar{\omega}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}. \tag{4}$$

And

$$D = \left[\frac{\bar{z}'(\bar{\zeta})}{z'(\zeta)}\right]^{\frac{1}{2}} D_0, \tag{5}$$

so that

$$8\mu D = [z'(\zeta)\bar{z}'(\bar{\zeta})]^{-\frac{1}{2}}[\kappa\Omega(\zeta)\bar{z}'(\bar{\zeta})-z(\zeta)\bar{\Omega}'(\bar{\zeta})-\bar{\omega}'(\bar{\zeta})]. \tag{6}$$

Relations between the complex potentials suitable for obtaining any desired conditions of stress or displacement along the boundary $\eta = 0$

Let

$$\Lambda(\zeta) = \bar{z}(\zeta)\frac{d}{d\zeta}\frac{\Omega'(\zeta)}{z'(\zeta)}, \qquad \Gamma(\zeta) = \int^{\zeta}\Lambda(\zeta)\,d\zeta. \tag{7}$$

(i) Let

$$\omega(\zeta) = -\int^{\zeta} z'(\zeta)\Gamma(\zeta)\,d\zeta \tag{8}$$

so that

$$\frac{d}{d\zeta}\frac{\omega'(\zeta)}{z'(\zeta)} = -\Lambda(\zeta) \tag{9}$$

and (3) becomes

$$4(\breve{\eta\eta}+i\breve{\xi\eta}) = \frac{\Omega'(\zeta)}{z'(\zeta)}+\frac{\bar{\Omega}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}+\frac{\bar{z}(\bar{\zeta})-\bar{z}(\zeta)}{\bar{z}'(\bar{\zeta})}\frac{d}{d\zeta}\frac{\Omega'(\zeta)}{z'(\zeta)}. \tag{10}$$

Thus

$$4(\breve{\eta\eta}+i\breve{\xi\eta})_{\eta=0} \equiv 4(\breve{\eta\eta}+i\breve{\xi\eta})^0 = 2\,\mathrm{re}\left[\frac{\Omega'(\xi)}{z'(\xi)}\right]. \tag{11}$$

From (11) we see that the combination gives zero shear over $\eta = 0$ and if we require

$$\breve{\eta\eta}^0 = f(\xi), \tag{12}$$

we must choose $\Omega(\zeta)$ so that

$$\mathrm{re}\left[\frac{\Omega'(\xi)}{z'(\xi)}\right] = 2f(\xi). \tag{13}$$

(ii) Let

$$\omega(\zeta) = \int^{\zeta}[z'(\zeta)\Gamma(\zeta)-2\bar{z}(\zeta)\Omega'(\zeta)]\,d\zeta \tag{14}$$

so that

$$\frac{d}{d\zeta}\frac{\omega'(\zeta)}{z'(\zeta)} = -\Lambda(\zeta)-2\bar{z}'(\zeta)\frac{\Omega'(\zeta)}{z'(\zeta)}, \tag{15}$$

and (3) becomes

$$4(\breve{\eta\eta}+i\breve{\xi\eta}) = \frac{\bar{\Omega}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}-\frac{\Omega'(\zeta)}{z'(\zeta)}\left[2\frac{\bar{z}'(\zeta)}{\bar{z}'(\bar{\zeta})}-1\right]+\frac{\bar{z}(\bar{\zeta})-\bar{z}(\zeta)}{\bar{z}'(\bar{\zeta})}\frac{d}{d\zeta}\frac{\Omega'(\zeta)}{z'(\zeta)}. \tag{16}$$

Thus
$$4(\breve{\eta\eta}+i\breve{\xi\eta})^0 = -2i\,\mathrm{im}\frac{\Omega'(\xi)}{z'(\xi)}. \tag{17}$$

From (17) we see that the combination gives zero normal pressure over $\eta = 0$ and if we require
$$\breve{\eta\eta}^0 = \phi(\xi),$$
we must choose $\Omega(\zeta)$ so that
$$\mathrm{im}\left[\frac{\Omega'(\xi)}{z'(\xi)}\right] = -2\phi(\xi). \tag{18}$$

(iii) Let
$$\omega(\zeta) = -\int^{\zeta}[\kappa\Omega(\zeta)\bar{z}'(\zeta)+\bar{z}(\zeta)\Omega'(\zeta)]\,d\zeta. \tag{19}$$
Then (6) becomes
$$8\mu D = [z'(\zeta)\bar{z}'(\bar{\zeta})]^{-\frac{1}{2}}[\kappa\Omega(\zeta)\bar{z}'(\bar{\zeta})+\kappa\bar{\Omega}(\bar{\zeta})z'(\bar{\zeta})+\{z(\bar{\zeta})-z(\zeta)\}\bar{\Omega}'(\bar{\zeta})]. \tag{20}$$
Thus
$$8\mu D^0 = 2\,\mathrm{re}[z'(\xi)\bar{z}'(\xi)]^{-\frac{1}{2}}[\kappa\Omega(\xi)\bar{z}'(\xi)]. \tag{21}$$
From (21) we see that the combination gives zero normal displacements over $\eta = 0$ and if we require
$$u^0 = \psi(\xi), \tag{22}$$
we must choose $\Omega(\zeta)$ so that
$$\mathrm{re}[z'(\xi)\bar{z}'(\xi)]^{-\frac{1}{2}}[\kappa\Omega(\xi)\bar{z}'(\xi)] = 4\mu\psi(\xi), \tag{23}$$
i.e.
$$\mathrm{re}[\kappa\Omega(\xi)\bar{z}'(\xi)] = [z'(\xi)\bar{z}'(\xi)]^{\frac{1}{2}}4\mu\psi(\xi). \tag{24}$$

(iv) Let
$$\omega(\zeta) = \int^{\zeta}[\kappa\Omega(\zeta)\bar{z}'(\zeta)-\bar{z}(\zeta)\Omega'(\zeta)]\,d\zeta. \tag{25}$$
Then (6) becomes
$$8\mu D = [z'(\zeta)\bar{z}'(\bar{\zeta})]^{-\frac{1}{2}}[\kappa\Omega(\zeta)\bar{z}'(\bar{\zeta})-\kappa\bar{\Omega}(\bar{\zeta})z'(\bar{\zeta})+\{z(\bar{\zeta})-z(\zeta)\}\bar{\Omega}'(\bar{\zeta})]. \tag{26}$$
Thus
$$8\mu D^0 = 2i\,\mathrm{im}[z'(\xi)\bar{z}'(\xi)]^{-\frac{1}{2}}[\kappa\Omega(\xi)\bar{z}'(\xi)]. \tag{27}$$
From (27) we see that the combination gives zero tangential displacements over $\eta = 0$ and if we require
$$v^0 = \chi(\xi), \tag{28}$$
we must choose $\Omega(\zeta)$ so that
$$\mathrm{im}[z'(\xi)\bar{z}'(\xi)]^{-\frac{1}{2}}[\kappa\Omega(\xi)\bar{z}'(\xi)] = 4\mu\chi(\xi), \tag{29}$$
i.e.
$$\mathrm{im}[\kappa\Omega(\xi)\bar{z}'(\xi)] = [z'(\xi)\bar{z}'(\xi)]^{\frac{1}{2}}4\mu\chi(\xi). \tag{30}$$
From equations (13), (18), (24), (30) we see that problems involving elastic material occupying the region $\eta \geqslant 0$ of the z-plane and subjected to specified stresses or displacements over $\eta = 0$ reduce to the determination of functions of ζ which are analytic in $\eta > 0$, of suitable orders of magnitude at

infinity, and which have specified real or imaginary parts when $\eta = 0$. As in the case of the half-plane we may show that if

$$F(\zeta) = \frac{1}{\pi}\int\limits_{0}^{\infty} e^{i\zeta u} f_T(u)\,du, \text{ where } f_T(u) = \int\limits_{-\infty}^{\infty} f(t)e^{-iut}\,dt, \tag{31}$$

then $F(\zeta)$ is analytic in $\eta > 0$ and $\operatorname{re} F(\xi) = f(\xi)$. Also $\operatorname{im} iF(\xi) = f(\xi)$. Thus, we may satisfy (13) by

$$\frac{\Omega'(\zeta)}{z'(\zeta)} = I(\zeta) = \frac{2}{\pi}\int\limits_{0}^{\infty} e^{i\zeta u} f_T(u)\,du; \tag{32}$$

(18) is satisfied by

$$\frac{\Omega'(\zeta)}{z'(\zeta)} = J(\zeta) = -\frac{2i}{\pi}\int\limits_{0}^{\infty} e^{i\zeta u} \phi_T(u)\,du. \tag{33}$$

We may satisfy (24), (30) by equations of similar forms.

Orders of magnitude at infinity

If $$\int\limits_{-\infty}^{\infty} |tf(t)|\,dt < \infty, \qquad \int\limits_{-\infty}^{\infty} |t\phi(t)|\,dt < \infty,$$

then $$I(\zeta),\ J(\zeta) = O(\zeta^{-1}) \text{ at infinity in } \eta > 0. \tag{34}$$

Thus, if $z(\zeta) = O(\zeta^s)$ at infinity ($s \neq 1$), it follows that $\Omega(\zeta) = O(\zeta^{s-1}) = o(z)$ and $\omega(\zeta) = O(\zeta^{2s-1}) = o(z^2)$ so that the stresses at infinity are evanescent. If $s = 1$, then $\Omega(\zeta) = O(\log\zeta)$ and $\omega(\zeta) = O(\zeta\log\zeta)$ and the above conclusion still holds.

Application of the above theory to material with a parabolic boundary

Consider the transformation

$$(\zeta + ia)^2 = 2iz. \tag{35}$$

The boundary $\eta = 0$ is given by the Cartesian equation

$$x^2 + 2a^2y = a^4 \tag{36}$$

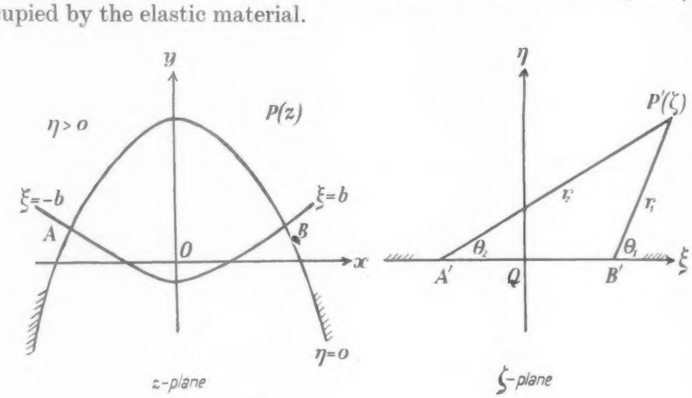
and the locus $\xi = \pm b$ is given by

$$x^2 - 2b^2y = b^4. \tag{37}$$

The curves given in (36), (37) intersect at the points $(\pm ab,\ (a^2 - b^2)/2)$ denoted by A and B in Fig. 2. Also

$$z'(\zeta) = a - i\zeta. \tag{38}$$

Thus $z'(\zeta)$ is zero at $\zeta = -ia$ but does not vanish in the region $\eta \geqslant 0$ occupied by the elastic material.



$$A = \left(-ab,\ \frac{a^2-b^2}{2}\right) \qquad A' = (-b,\ 0)$$

$$B = \left(ab,\ \frac{a^2-b^2}{2}\right) \qquad B' = (+b,\ 0)$$

FIG. 2.

We now apply the above theory to the determination of stresses and displacements in elastic material which occupies the region $\eta \geqslant 0$ and is subjected to specified stresses over $\eta = 0$.

(i) Given $$\breve{\eta\eta}^0 = f_1(\xi) = \begin{cases} -1, & |\xi| < b, \\ 0, & |\xi| > b, \end{cases} \qquad \breve{\xi\eta}^0 = 0. \tag{39}$$

From (32) $$I_1(\zeta) = \frac{2i}{\pi}\log\frac{\zeta-b}{\zeta+b} \tag{40}$$

and $$\Omega_1'(\zeta) = \frac{2}{\pi}(\zeta+ia)\log\frac{\zeta-b}{\zeta+b}, \tag{41}$$

whence

$$\pi\Omega_1(\zeta) = [b-(\zeta+2ia)](\zeta+b)\log(\zeta+b)+ \\ +[b+(\zeta+2ia)](\zeta-b)\log(\zeta-b)-2b\zeta. \tag{42}$$

From (7) $$\pi\Lambda_1(\zeta) = \frac{(b+ia)^2}{\zeta+b}-\frac{(b-ia)^2}{\zeta-b}-2b \tag{43}$$

and $$\pi\Gamma_1(\zeta) = (b+ia)^2\log(\zeta+b)-(b-ia)^2\log(\zeta-b)-2b\zeta, \tag{44}$$

whence

$$2\pi\omega_1(\zeta) = \tfrac{2}{3}[-2ib\zeta^3+6ab\zeta^2-9ia^2b\zeta+3ib^3\zeta]+ \\ +[2a-i\zeta-ib](b-ia)^2(\zeta-b)\log(\zeta-b)- \\ -[2a-i\zeta+ib](b+ia)^2(\zeta+b)\log(\zeta+b). \tag{45}$$

(ii) Given $$\breve{\eta\eta}^{0} = f_2(\xi) = \begin{cases} -\xi^2, & |\xi| < b, \\ 0, & |\xi| > b, \end{cases} \qquad \breve{\xi\eta}^{0} = 0. \tag{46}$$

From (32) $$I_2(\zeta) = \frac{2i}{\pi}\left[2b\zeta+\zeta^2\log\frac{\zeta-b}{\zeta+b}\right] \tag{47}$$

and $$\Omega_2'(\zeta) = \frac{2i}{\pi}\left[2ab\zeta-2ib\zeta^2+a\zeta^2\log\frac{\zeta-b}{\zeta+b}-i\zeta^3\log\frac{\zeta-b}{\zeta+b}\right],$$

whence

$$6\pi\Omega_2(\zeta) = 2i[4ab\zeta^2+3ib\zeta(b^2-\zeta^2)]+[3(\zeta^4-b^4)+4ia(\zeta^3-b^3)]\log(\zeta-b)- \\ -[3(\zeta^4-b^4)+4ia(\zeta^3+b^3)]\log(\zeta+b). \tag{48}$$

From (7)

$$\pi\Lambda_2(\zeta) = -4b\zeta^2+8iab\zeta+2a^2b-2b^3+ \\ +\frac{b^4+2iab^3}{\zeta+b}-\frac{b^4-2iab^3}{\zeta-b}-[2\zeta^3-4ia\zeta^2-2a^2\zeta]\log\frac{\zeta-b}{\zeta+b} \tag{49}$$

and

$$6\pi\Gamma_2(\zeta) = -6b\zeta^3+16iab\zeta^2-6b^3\zeta+3\zeta^4\log\frac{\zeta+b}{\zeta-b}-8ia\zeta^3\log\frac{\zeta+b}{\zeta-b}- \\ -6a^2\zeta^2\log\frac{\zeta+b}{\zeta-b}+(3b^4+6a^2b^2)\log\frac{\zeta+b}{\zeta-b}+4iab^3\log(\zeta^2-b^2), \tag{50}$$

whence

$$12\pi\omega_2(\zeta) \\ = \tfrac{2}{3}[-3ib\zeta^5-6ab\zeta^4-(3a^2+5b^2)ib\zeta^3+6ab(3b^2-2a^2)\zeta^2+3ib^3(4b^2+3a^2)\zeta]+ \\ +[i(\zeta^6-b^6)+2a(\zeta^5+b^5)+ia^2(\zeta^4-b^4)+4a^3(\zeta^3+b^3)+ \\ +b^3(3ib-4a)(\zeta^2-b^2)-2ab^3(3b+4ia)(\zeta+b)]\log(\zeta+b)- \\ -[i(\zeta^6-b^6)+2a(\zeta^5-b^5)+ia^2(\zeta^4-b^4)+4a^3(\zeta^3-b^3)+ \\ +b^3(3ib+4a)(\zeta^2-b^2)-2ab^3(3b-4ia)(\zeta-b)]\log(\zeta-b). \tag{51}$$

(iii) Given $$\breve{\eta\eta}^{0} = \begin{cases} \xi^2-b^2, & |\xi| < b, \\ 0, & |\xi| > b, \end{cases} \qquad \breve{\xi\eta}^{0} = 0. \tag{52}$$

This is the problem of hydrostatic loading up to the level AB in Fig. 1, the solution being interpreted in terms of plane strain. The complex potentials are given by a linear combination of the two preceding pairs of potentials, viz. $$\Omega_3(\zeta) = b^2\Omega_1(\zeta)-\Omega_2(\zeta), \qquad \omega_3(\zeta) = b^2\omega_1(\zeta)-\omega_2(\zeta). \tag{53}$$

From (1), (2)

$$2\Theta = \frac{4}{\pi}(\xi^2-\eta^2-b^2)(\theta_1-\theta_2)+\frac{8\eta}{\pi}\left(b+\xi\log\frac{r_1}{r_2}\right) \quad \text{(see Fig. 1)}, \tag{54}$$

$$-2\overline{\Phi} = -\frac{8\eta}{\pi}\frac{\xi^2+a\eta+a^2+i\xi\eta}{(a+\eta)^2+\xi^2}\left\{\left[2b+\xi\log\frac{r_1}{r_2}-\eta(\theta_1-\theta_2)\right]+ \right. \\ \left. +i\left[\eta\log\frac{r_1}{r_2}+\xi(\theta_1-\theta_2)\right]\right\}, \tag{55}$$

whence

$$\pi\widetilde{\xi\eta}[(a+\eta)^2+\xi^2]$$

$$= -2\eta\left[2b\xi\eta+\eta(2\xi^2+a^2+a\eta)\log\frac{r_1}{r_2}+\xi(\theta_1-\theta_2)(\xi^2-\eta^2+a^2+a\eta)\right], \quad (56)$$

and

$$\pi\widetilde{\eta\eta}[(a+\eta)^2+\xi^2] = 2b\eta(\eta^2-\xi^2-a^2)+2\xi\eta^2(a+2\eta)\log\frac{r_1}{r_2}+$$

$$+(\theta_1-\theta_2)[(\xi^2-b^2)(\xi^2+a^2+2a\eta+4\eta^2)-\eta^2(\eta^2+a^2-3b^2)], \quad (57)$$

and

$$\widetilde{\xi\xi} = \Theta-\widetilde{\eta\eta}. \quad (58)$$

The stresses are finite and uniform in $\eta \geqslant 0$ and are evanescent at infinity. The displacements may be evaluated from (4). We write

$$8\mu D_0^{(1)} = \kappa\Omega(\zeta), \qquad 8\mu D_0^{(2)} = -z(\zeta)\frac{\overline{\Omega}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}-\frac{\bar{\omega}'(\bar{\zeta})}{\bar{z}'(\bar{\zeta})}. \quad (59)$$

From (42), (48) we see that $D_0^{(1)}$ is uniform and continuous in $\eta \geqslant 0$. From (40), (44), (47), (50)

$$8\mu\overline{D}_0^{(2)} = \frac{2b\zeta}{\pi}(\bar{\zeta}-ia)^2-\frac{b\zeta}{3\pi}[3\zeta^2-8ia\zeta-3b^2-6a^2]-$$

$$-\frac{1}{6\pi}[6(\bar{\zeta}-ia)^2(\zeta^2-b^2)-3(\zeta^4-b^4)+8ia(\zeta^3+b^3)+6a^2(\zeta^2-b^2)]\log(\zeta+b)+$$

$$+\frac{1}{6\pi}[6(\bar{\zeta}-ia)^2(\zeta^2-b^2)-3(\zeta^4-b^4)+8ia(\zeta^3-b^3)+6a^2(\zeta^2-b^2)]\log(\zeta-b). \quad (60)$$

Thus $D_0^{(2)}$ is uniform and continuous in $\eta \geqslant 0$. Equations similar in form to (56)–(58) were given by Conrad (**3**). His solution to the problem is, however, not valid as the integrals chosen for complex potentials are divergent and the mathematical operations on such integrals are not permissible. The integrands may be modified as done by Filon (**4**) when dealing with the infinite strip. There is, however, no need to work with potentials in the form of integrals as the problem requires only elementary functions. We have contented ourselves with obtaining solutions in which the stresses at infinity are evanescent without specifying orders of magnitude at infinity. We now show that the solution is not unique. From Stevenson (**1**) the condition that a boundary is stress-free is that

$$\overline{\Omega}(\bar{z})+\bar{z}\Omega'(z)+\omega'(z) = \text{constant} \quad (61)$$

along the boundary. Thus, taking the constant to be zero, in the present case the boundary $\eta = 0$ is stress-free if

$$2\overline{\Omega}(\zeta)(\zeta+ia)-\Omega'(\zeta)(\zeta-ia)^2+2i\omega'(\zeta) = 0. \quad (62)$$

Equation (62) is satisfied by the potentials

$$\Omega_0(\zeta) = \zeta, \qquad \omega_0(\zeta) = \frac{i\zeta}{6}(\zeta^2+15a^2). \tag{63}$$

These potentials give evanescent stresses at infinity and may be superimposed on the above solutions without violating the boundary conditions. If we put $\Omega(\zeta) = \zeta$ in (8) the term in $\omega(\zeta)$ of highest degree is $i\zeta^3/6$. Hence (63) may be employed to remove the term in $\Omega(\zeta)$ of order ζ at infinity and that in $\omega(\zeta)$ of order ζ^3 from all solutions obtained from (8). The stresses given by (63) are

$$2\widetilde{\xi\xi}[(\eta+a)^2+\xi^2]^2 = (\eta+a)(2\xi^2+2a^2+2a\eta+\eta^2), \tag{64}$$

$$2\widetilde{\eta\eta}[(\eta+a)^2+\xi^2]^2 = \eta(\eta+a)(\eta+2a), \tag{65}$$

$$2\widetilde{\xi\eta}[(\eta+a)^2+\xi^2]^2 = \xi\eta(\eta+2a). \tag{66}$$

In all the solutions obtained above the stresses at infinity are $O(\zeta^{-1})$, i.e. $O(z^{-\frac{1}{2}})$ (see equation (34)). We may use the potentials given in (63) to obtain solutions to the problems defined in (39), (46), (52) with stresses at infinity which are $O(z^{-1})$. For large $|\zeta|$ (42), (45) become

$$\Omega_1(\zeta) = -4b\zeta/\pi+O(\log\zeta), \qquad \omega_1(\zeta) = -2ib\zeta^3/3\pi+O(\zeta^2\log\zeta). \tag{67}$$

The required modified potentials $\Omega_{1,m}(\zeta)$, $\omega_{1,m}(\zeta)$ are

$$\Omega_{1,m}(\zeta) = \Omega_1(\zeta)+\frac{4b}{\pi}\Omega_0(\zeta), \qquad \omega_{1,m}(\zeta) = \omega_1(\zeta)+\frac{4b}{\pi}\omega_0(\zeta). \tag{68}$$

Similarly, for large $|\zeta|$ (48), (51) become

$$\Omega_2(\zeta) = -4b^3\zeta/3\pi+O(\log\zeta), \qquad \omega_2(\zeta) = -2ib^3\zeta^3/9\pi+O(\zeta^2\log\zeta). \tag{69}$$

The required modified potentials in this case are

$$\Omega_{2,m}(\zeta) = \Omega_2(\zeta)+\frac{4b^3}{3\pi}\Omega_0(\zeta), \qquad \omega_{2,m}(\zeta) = \omega_2(\zeta)+\frac{4b^3}{3\pi}\omega_0(\zeta). \tag{70}$$

Thus the modified form of (53) is

$$\Omega_{3,m}(\zeta) = \Omega_3(\zeta)+\frac{8b^3}{3\pi}\Omega_0(\zeta), \qquad \omega_{3,m}(\zeta) = \omega_3(\zeta)+\frac{8b^3}{3\pi}\omega_0(\zeta). \tag{71}$$

The modifications are equivalent to superimposing multiples of the stresses given in (64)–(66) on to the stress systems given by the general method of solution. The additional stresses have a profound effect on the stresses in the finite part of the material even though they do not affect the boundary conditions over $\eta = 0$ and give evanescent stresses at infinity. The solution emphasizes the importance of specifying stress components at infinity other than those which are $O(z^{-1})$. In the above example the orders of magnitude of the modified potentials are the lowest possible since the stresses applied

along $\eta = 0$ have a non-zero force resultant. Their uniqueness (**5**) may be shown by a method similar to that given for the half-plane (**6**).

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ON THE PROBABILITY FUNCTION IN A NORMAL MULTIVARIATE DISTRIBUTION†

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1. An integral occurring in Probability Theory is

$$Z_n(z,a) = \int\limits_0^z \int\limits_0^\pi \frac{t(t\sin\theta)^{n-2}}{\Gamma(\frac{1}{2}n-\frac{1}{2})2^{\frac{1}{2}n-1}\sqrt{\pi}} \exp\{-\tfrac{1}{2}(t^2+a^2-2at\cos\theta)\}\, d\theta dt.$$

This corresponds to the non-central Chi-square Distribution ($a \neq 0$) and represents the probability (**1**) that an observation $(x_1, x_2,..., x_n)$ falls within a given distance σz of any point $P(a_1, a_2,..., a_n)$, the origin being at the point O, where

$$OP = \sigma a \ \{= \sqrt{(\textstyle\sum a_i^2)}\}.$$

An interesting application of this in Ballistics is to the problem of estimating the relative risks at different distances from an aiming point when each projectile has a zone of risk.

In any specific problem, the value of $Z_n(z,a)$ for small values of z and a will be required. The object of this note is to give a few relevant formulae.

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2. Since

$$e^{ax-\frac{1}{2}a^2} = \sum_{r=0}^{\infty} \frac{a^r}{r!} H_r(x) \tag{2.1}$$

where $H_r(x)$ is the Hermite polynomial defined by

$$H_r(x) = \sum_{j=0}^{j\leq\frac{1}{2}r} \frac{(-)^j(-r,2j)}{2^j(1,j)} x^{r-2j}. \tag{2.2}$$

The function $\exp\{-\frac{1}{2}(t^2+a^2-2at\cos\theta)\}$ can be expanded in powers of t and also in powers of a.

If we consider the first alternative we get

$$Z_n(z,a) = \int\limits_0^z \int\limits_0^\pi \frac{t(t\sin\theta)^{n-2}e^{-\frac{1}{2}a^2}}{\Gamma(\frac{1}{2}n-\frac{1}{2})2^{\frac{1}{2}n-1}\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{t^r}{r!} H_r(a\cos\theta)\, d\theta dt.$$

† My thanks are due to the referee for advising me to rewrite the introductory section 1 of this paper.

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Term-by-term integration with respect to t is obviously permissible, and hence

$$Z_n(z,a) = \frac{e^{-\frac{1}{2}a}}{\Gamma(\frac{1}{2}n-\frac{1}{2})2^{\frac{1}{2}n-1}\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{z^{r+n}}{r!\,(r+n)} \int_0^{\pi} \sin^{n-2}\theta\, H_r(a\cos\theta)\,d\theta$$

$$= \frac{e^{-\frac{1}{2}a^2}}{2^{\frac{1}{2}n-1}\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{z^{2r+n}}{(2r)!\,(2r+n)} \sum_{j=0}^{j=r} \frac{(-)^j(-2r,2j)\Gamma(r-j+\frac{1}{2})}{2^j(1,j)\Gamma(r-j+\frac{1}{2}n)}\, a^{2r-2j},$$

since

$$\int_0^{\pi} \sin^m\theta \cos^n\theta\, d\theta = 0 \quad \text{when } n \text{ is odd,}$$

$$= \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}m+\frac{1}{2}n+1)}, \quad \text{otherwise.}$$

3. The other alternative, contemplated in section 2, gives

$$Z_n(z,a) = \int_0^z \int_0^{\pi} \frac{t(t\sin\theta)^{n-2}e^{-\frac{1}{2}t^2}}{\Gamma(\frac{1}{2}n-\frac{1}{2})2^{\frac{1}{2}n-1}\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{a^r}{r!} H_r(t\cos\theta)\,d\theta dt.$$

Since
$$Z_n(z,0) = \frac{1}{2^{\frac{1}{2}n-1}\Gamma(\frac{1}{2}n)} \int_0^z t^{n-1}e^{-\frac{1}{2}t^2}\,dt,$$

we get the expansion

$$Z_n(z,a) = \sum_{r=0}^{\infty} \frac{a^{2r}}{(2r)!} \sum_{j=0}^{j=r} \frac{(-)^j(-2r,2j)}{(1,j)}\, 2^{r-2j}\Gamma(r-j+\tfrac{1}{2}) Z_{2r-2j+n}(z,0).$$

4. An expansion similar to the one in section 3 is obtained by expanding $\exp(at\cos\theta)$ and integrating term by term. We then have

$$Z_n(z,a) = \exp(-\tfrac{1}{2}a^2) \sum_{r=0}^{\infty} \frac{a^{2r}}{2^r\Gamma(r+1)} Z_{n+2r}(z,0),$$

where we have used the duplication formula for the gamma function.

5. An interesting reciprocal formula will now be obtained for the function $Z_n(z,0)$, the probability corresponding to the central Chi-square Distribution. Taking $|h|$ to be small, we have

$$\sum_{n=0}^{\infty} h^n Z_{n+1}(z,0) \frac{2^{\frac{1}{2}n-\frac{1}{2}}\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(n+1)} = \int_0^z e^{-\frac{1}{2}t^2+th}\,dt = \sum_{n=0}^{\infty} \frac{z^{n+1}}{\Gamma(n+2)} H_n(h)$$

true for small values of z.

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DIFFRACTION BY AN EDGE AND BY A CORNER

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SUMMARY

Conditions are given which are sufficient to ensure that the current density normal to an edge is zero at the edge and that there is no line distribution of charge on the edge. An extra condition is given which makes the components of the field parallel to the edge finite. The solution is then shown to be unique. Simpler conditions are given for two-dimensional fields. The agreement of various known solutions with our assumptions is discussed. Finally it is shown that certain simple current densities lead to a unique solution for the diffraction by corners of flat surfaces.

Introduction

BOUWKAMP (**1**) has indicated how one may obtain any number of solutions to the problem of the two-dimensional diffraction of plane harmonic electromagnetic waves by a perfectly conducting semi-infinite plane by taking derivatives of any known solution. Two different solutions for the three-dimensional diffraction by a semi-infinite plane have been given by Wiegrefe (**2**) and by the author (**3**). Various solutions have been given for the diffraction by a circular disk; two different ones are those of Möglich (**4**) and Meixner and Andrejewski (**5**). As a consequence it has become necessary to impose conditions that ensure a unique solution.

The question has been discussed by Bouwkamp (**6**), Maue (**7**), Meixner (**8**), and the author (**3**). The results of all authors are in agreement although their viewpoints are different. In all cases the edge has been taken to be a curve with continuously turning tangent and apart from Maue's work the edge has been on an (essentially plane) infinitely thin conductor. Bouwkamp has argued that the field and currents must behave near the edge as they do in the Sommerfeld two-dimensional solution for the semi-infinite plane. Meixner uses the fact that near the edge the fields are quasi-stationary and the assumption that the integral of the electromagnetic energy over any region must be finite to deduce the forms of the fields. It is then possible to deduce the currents and charges; they agree with those found by Maue using an integral equation formulation. The author attempted to show that the assumption of finite current and charge on the conductor gives a unique field and that the current perpendicular to the edge vanishes, but the proof is not completely rigorous.

It follows from the work of Meixner and the author that, whether or not the bounding curve of an infinitely thin surface is plane, a sufficient condition for a unique solution (apart from radiation conditions at infinity) is that the integral of the Poynting vector over a small cylinder surrounding the edge should tend to zero as the radius δ of the cylinder tends to zero i.e.

$$\lim_{\delta \to 0} \int_{S_1} (\mathbf{E} \wedge \mathbf{B}^* + \mathbf{E}^* \wedge \mathbf{B}) . d\mathbf{S} = 0,$$

where S_1 is the cylindrical surface and the asterisk denotes a complex conjugate.

For the actual solution of problems it is necessary to reduce this condition to simpler forms of various kinds. If the problem is formulated in terms of the field it is sufficient that the components of $\mathbf{E}$ and $\mathbf{B}$ parallel to the edge should both be $O(1)$ and that the remaining components should be $o(\delta^{-1})$. In two-dimensional diffraction when the fields depend only upon E_x or B_x the integration is round a small circle and sufficient conditions are that $E_x = O(1)$ or $B_x = O(1)$ in the two cases, as has been indicated by the author (**9**). Frequently, however, the problem is formulated in terms of integral equations involving the current and charge densities. In this case one wishes to know firstly whether line integrals along the edge are necessary and, secondly, conditions on the current and charge densities to ensure that the solution to the diffraction problem is unique.

In this paper we are concerned with the integral equation approach and the analysis is kept fairly general; the curvature of the surface is not neglected except in section 9. The work is complicated by the fact that we have to allow the charge and current densities to be infinite at the edge since all known solutions have singularities at the edge. In section 1 there is a brief discussion of the validity of the integral expressions. Section 2 shows that the singularities (if any) of the field are derived from the intensities and potentials of certain static charge distributions. Section 3 is devoted to enumerating the conditions imposed on the current and charge densities and the reasons for their choice. As a consequence of these conditions we show in section 4 that the current density normal to the edge is zero. In section 5 we give sufficient conditions for the field components parallel to the edge to be finite. It is shown in section 6 that as a consequence of these restrictions the field is unique. Section 7 deals with the modifications for two-dimensional fields and section 8 indicates how some known solutions agree with our restrictions. Finally, we discuss the uniqueness of certain current distributions in the corners of flat surfaces.

1. The validity of the integral expressions for the field

We shall use the definitions of regular surfaces and regular curves that are given by Kellogg (see **10**, ch. 4) with some restrictions. Thus if $z = f(x, y)$, where (x, y) lies in a regular region of the (x, y)-plane, is the representation of the set of points forming a regular surface element we shall assume that f is continuous and has continuous second partial derivatives bounded in absolute value by some constant M. The (x, y)-plane will usually be the tangent plane. If $y = g_1(x)$, $z = g_2(x)$ is the representation of a regular arc, we shall assume that g_1 and g_2 are continuous and have continuous second derivatives bounded in an absolute value by M. Apart from this, in this section we shall assume that the regular surface has a continuously turning normal and that the regular curve has a continuously turning tangent except when specifically stated otherwise.

Let S be an unclosed perfectly conducting two-sided regular surface surrounded by free space. Let the field whose electric intensity is $\mathbf{E}_0$ and whose magnetic flux density is $\mathbf{B}_0$ be incident upon S. We use m.k.s. units and suppress the time factor $e^{i\omega t}$. The incident field excites a surface current density $\mathbf{I}$ and a surface charge density σ on S which satisfy the continuity equation $\operatorname{div} \mathbf{I} + i\omega\sigma = 0$. There may also be present a line charge of linear density σ_1 on C, the bounding curve of S. Then a solution of Maxwell's equations (under certain conditions) is

$$\left.\begin{aligned} \mathbf{E}_1 &= \mathbf{E}_0 - \frac{i\omega}{4\pi k^2}\int\limits_S \{k^2\psi\mathbf{I} + i\omega\sigma \operatorname{grad}\psi\}\, dS - \frac{i\omega}{4\pi k^2}\oint\limits_C \sigma_1 \operatorname{grad}\psi\, ds \\ \mathbf{B}_1 &= \mathbf{B}_0 + \frac{1}{4\pi}\int\limits_S \{\mathbf{I}\wedge \operatorname{grad}\psi\}\, dS \end{aligned}\right\} \tag{1}$$

where $\mathbf{E}_1$, $\mathbf{B}_1$ are the electric intensity and magnetic flux density at the point of observation (x_1, y_1, z_1), $k = 2\pi/(\text{wave-length})$, $\psi = e^{-ikr}/r$ and

$$r^2 = (x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2,$$

(x, y, z) being a point of S or C. The current density $\mathbf{I}$ must lie in S and therefore satisfies $\mathbf{I}.\mathbf{n} = 0$, where $\mathbf{n}$ is a unit vector along the normal to S, being chosen so that it is always directed from the same side of S.

In order that the integral expressions may have a meaning when the point of observation is in free space we assume that $\mathbf{I}$ and σ are summable (Lebesgue) on S. Neither of these assumptions implies the other. The assumption implies that $\mathbf{I}$ and σ are finite almost everywhere on S. We shall assume that the only place where finiteness and continuity may fail is on C. If part of the surface goes to infinity a difficulty arises because a summable quantity is also absolutely summable and this may not be true for the current and charge densities occurring in practice. The difficulty can usually

be overcome by taking the surrounding medium to be slightly conducting until after the integrals have been evaluated. Another device which is sometimes successful is to define $\int^{\infty}$ as $\lim_{N\to\infty}\int^{N}$. For simplicity we shall assume in the following that S is finite in area and C is finite in length. Also, for the line integral in (1) to have a meaning, we require σ_1 to be summable on C and therefore finite at almost all points of C. We assume that the finiteness of σ_1 can fail only at points of C where the direction of C changes discontinuously.

The method of verifying that $\mathbf{E}_1$, $\mathbf{B}_1$ satisfy Maxwell's equations at points of free space is well known (see, for example, Stratton (**11**)). The verification requires firstly that it shall be permissible to take derivatives under the integral sign, secondly that Stokes's theorem can be applied to $\oint_C \sigma_1 \psi\, ds$ and $\oint_C \sigma_1 \operatorname{grad}\psi\, ds$, and thirdly that $\sigma_1 = \mathbf{n}\wedge\mathbf{I}.\mathbf{t}$ on C, where $\mathbf{t}$ is a unit vector tangent to C, its direction being determined from $\mathbf{n}$ by the right-hand screw law.

There is no difficulty about the taking of derivatives since the integrand is the sum of products of a summable function and a continuous function when the point of observation is not on S or C so that the integrals exist.

As regards the third step, the remark already made about σ_1 implies that $\mathbf{n}\wedge\mathbf{I}.\mathbf{t}$ is finite at all points of C where the tangent is continuously turning, i.e. that the current normal to C is finite.

Stokes's theorem for a two-sided regular surface can be proved whenever Green's formula in two dimensions holds for a regular region of the plane which is bounded by the projection of the boundary of the surface, i.e. when

$$\int_I \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) dxdy = \int_J (M\, dy + N\, dx),$$

where J is a closed regular curve and I its interior.

In most proofs (see, for example, Verblunsky (**12**)) it is required that M and N shall be continuous on $I+J$. This restriction is too heavy for our purposes since we do not wish to assume that all components of the current density are finite on C, although we may take them to be finite and continuous on S. We can, however, easily extend the theorem to cover our case. If $\partial M/\partial x$ and $\partial N/\partial y$ are summable, if M is the indefinite integral of $\partial M/\partial x$ with respect to x, if N is the indefinite integral of $\partial N/\partial y$, and M and N are continuous on I, then Green's formula holds for J_m, where J_m is a closed regular curve in I. In particular, this is true when J_m is a member of the sequence $\{J_m\}$ of simple closed regular curves $x = x_m(t)$, $y = y_m(t)$ on $I+J$

such that $\lim_{m\to\infty} x_m(t) = x(t)$, $\lim_{m\to\infty} y_m(t) = y(t)$ uniformly for $0 \leqslant t \leqslant l$, where $x = x(t)$, $y = y(t)$ $(0 \leqslant t \leqslant l)$ is J. Hestenes (**13**) has shown that this sequence exists when J is a closed rectifiable curve. Let M_m, N_m be the values of M and N on J_m and define $My'+Nx'$ as $\lim_{m\to\infty}(M_m y'_m+N_m x'_m)$ where primes indicate derivatives with respect to t. Now the measure of $I-I_m$ tends to zero as $m \to \infty$ and hence

$$\lim_{m\to\infty} \int\limits_{I_m} \left(\frac{\partial M}{\partial x}-\frac{\partial N}{\partial y}\right) dxdy = \int\limits_{I} \left(\frac{\partial M}{\partial x}-\frac{\partial N}{\partial y}\right) dxdy.$$

Thus Green's formula with $M\,dy+N\,dx$ replaced by $(My'+Nx')\,dt$ holds whenever we can assert that

$$\lim_{m\to\infty} \int\limits_{J_m} (M_m y'_m+N_m x'_m)\,dt = \int\limits_{J} (My'+Nx')\,dt.$$

Some conditions which are sufficient are (i) $|M_m y'_m+N_m x'_m| \leqslant K$ for all m and almost all t, where K is independent of m and t, or (ii) $\{M_m y'_m+N_m x'_m\}$ monotone and converging to $My'+Nx'$ with both $My'+Nx'$ and $M_1 y'_1+N_1 x'_1$ summable, or (iii) $\{M_m y'_m+N_m x'_m\}$ converges uniformly to $My'+Nx'$ at almost all points of C. In most physical situations the assumption (ii) will be satisfied; it requires that the current density perpendicular to the edge is finite almost everywhere on the edge and monotone in the neighbourhood, and we have already assumed the first part of this in connexion with the third step. In the particular case when J has a continuously turning tangent we may take J_m to be $x(t)-m^{-1}y'(t)$, $y(t)+m^{-1}x'(t)$. For large enough m the equations $x_m(t) = x_m(t_0)$, $y_m(t) = y_m(t_0)$ will have no solution other than $t = t_0$. Then, if $M_m = o(m)$, $N_m = o(m)$ as $m \to \infty$,

$$My'+Nx' = \lim_{m\to\infty}(M_m y'+N_m x')$$

so that if $My'+Nx'$ is finite on J, (ii) is satisfied when the sequence is monotone. We shall assume that one of the conditions given can always be complied with.

Accepting (1) as the expression for the field we obtain three integral equations for the determination of $\mathbf{I}$ and σ from the boundary conditions that the tangential components of $\mathbf{E}_1$ and the normal component of $\mathbf{B}_1$ vanish on S. Only two of these integral equations are independent.

2. The reduction of the fields due to the surface distribution

Write (1) as $\mathbf{E}_1 = \mathbf{E}_0+\mathbf{E}+\mathbf{E}'$, $\mathbf{B}_1 = \mathbf{B}_0+\mathbf{B}$ where

$$\mathbf{E}' = -(i\omega/4\pi k^2) \oint\limits_C \sigma_1 \operatorname{grad} \psi \, ds.$$

Now
$$|\psi - r^{-1}| = |r^{-1}(e^{-ikr}-1)| = O(1),$$
$$|\operatorname{grad}\psi - \operatorname{grad} r^{-1}| = |ikr^{-1}(e^{-ikr}-1)+r^{-2}(e^{-ikr}+ikr-1)||\operatorname{grad} r| = O(1)$$
and hence
$$\left.\begin{aligned} \mathbf{E} &= -\frac{i\omega}{4\pi k^2}\int_S \{k^2 r^{-1}\mathbf{I}+i\omega\sigma \operatorname{grad} r^{-1}\}\, dS+O(1) \\ \mathbf{B} &= \frac{1}{4\pi}\int_S \{\mathbf{I}\wedge \operatorname{grad} r^{-1}\}\, dS+O(1) \end{aligned}\right\}. \tag{2}$$
In particular
$$E_y = -\frac{i\omega}{4\pi k^2}\int_S \{k^2 r^{-1} I_y + i\omega\sigma(y-y_1)r^{-3}\}\, dS+O(1),$$
where the suffixes x, y, z denote components parallel to the coordinate axes. Thus the part of E_y which is possibly non-finite is the sum of the potential due to a surface charge distribution whose density is proportional to I_y and the y-component of the intensity of a surface charge density proportional to σ. In fact it is clear from (2) that $\mathbf{E}$ and $\mathbf{B}$ can be expressed in terms of the intensities and potentials of certain charge distributions. Therefore, if σ and $\mathbf{I}$ satisfy appropriate conditions, the theorems obtained by the author (**14**) can be applied. The conditions and the reasons for their choice are given in the next section.

3. The conditions imposed on the charge and current densities

We restate conditions B given in (**14**). They are that

(i) σ is summable on S.

(ii) $|\sigma-\sigma_0| \leqslant A(\epsilon)r_1^{\alpha}$, where σ_0 is the density at P of S, σ the density at a point distant r_1 from P, ϵ the shortest distance of P from C, and $A(\epsilon)$ and α are constants and $0 < \alpha \leqslant 1$. The condition is satisfied uniformly on S, except near C, where the condition holds in $r_1 < a\epsilon$, $0 < a < 1$, and $A(\epsilon)$ is constant in $r_1 < a\epsilon$ but such that
$$A(\epsilon) = o(\epsilon^{-1-\alpha}) \text{ as } \epsilon \to 0.$$

(iii) $|\sigma_0| = o(\epsilon^{-1})$ as $\epsilon \to 0$.

(iv) The line integral of σ along any line on S, whose points in comm with C (if any) form a set of (linear) measure zero, is finite.

It will be assumed that $\mathbf{I}$ also satisfies the same conditions with σ, A, α replaced by $\mathbf{I}$, A', α', where $0 < \alpha' \leqslant 1$ and $A' = o(\epsilon^{-1-\alpha'})$. These conditions will be known as B'. Since $\mathbf{I}$ satisfies B' the separate components of $\mathbf{I}$ also satisfy B'.

The reason for the choice of the modified Hölder continuity conditions B(ii) and B'(ii) is that the continuity alone of the charge density is insufficient to ensure the existence of the tangential components of the intensity on S. This is shown by a counter-example given by Kellogg (**10**). In practice, of course, $\mathbf{I}$ and σ will be differentiable on S and we can take $\alpha = 1$, $\alpha' = 1$. The modification of allowing A and A' to become infinite as C is approached is introduced in order to allow $\mathbf{I}$ and σ to become infinite on C. If this were not permitted all known solutions to diffraction problems would be excluded. We may note, however, that the component of $\mathbf{I}$ normal to C can be regarded as satisfying a Hölder condition with finite A' since we have already assumed that component to be finite on C. The use of Stokes's theorem in section 1 is also justified.

Conditions B(iv) and B'(iv) result from a consideration of the case when S is plane and C is a straight line. There we can expect variations parallel to C to be of no importance and that the charge per unit length parallel to C is finite, i.e. the integration of σ perpendicular to C gives a finite result.

Conditions B(iii) and B'(iii) are virtually implied by conditions B(iv) and B'(iv) in any physical problem. Conditions B(i) and B'(i) express the fact that the total current and charge on S are finite. Thus the conditions imposed, although appearing somewhat artificial at first sight, do in fact have a physical basis. It may be assumed in the actual solution of problems that conditions B(iv) and B'(iv) will be dominating ones. In a later section we see how some known solutions agree with our conditions.

It may be noted that it is not possible to deduce immediately restrictions on the field from those imposed on the current and charge densities since these densities are related only to the discontinuities of certain field components across S.

4. The consequences of the conditions imposed

Let O be a point of C in a neighbourhood of which C has a continuously turning tangent. Let $Oxyz$ be a system of axes with Oxy as the tangent plane to S at O and Ox as the tangent to C at O. Let $PXYZ$ be a system of axes with PXY as the tangent plane to S at P and such that $PXYZ$ becomes $Oxyz$ when P coincides with O. Then, since σ and the components of $\mathbf{I}$ satisfy conditions B and B', Theorem 1 of (**14**) shows that, at P,

$$|E_y| = o(\epsilon^{-1} + |\log \epsilon|), \qquad |\mathbf{E}| = o(\epsilon^{-1} + |\log \epsilon|), \quad |\mathbf{B}| = o(\epsilon^{-1})$$

as $\epsilon \to 0$, where $\epsilon = PO$. Therefore

$$|E_Y| = o(\epsilon^{-1}). \tag{3}$$

Also since the component of $\mathbf{I}$ normal to C satisfies a Hölder condition with finite A', so does σ_1, i.e. σ_1 satisfies conditions B_1, namely

(i) σ_1 is summable on C and

(ii) $|\sigma_1 - \sigma_{10}| \leqslant A_1 r_3^\gamma$, where A_1 and γ are constants, $0 < \gamma \leqslant 1$, σ_{10} is the density at O and r_3 is the distance between O and the point of C where σ_1 is the density.

One of our boundary conditions is that $E_{0Y} + E_Y + E'_Y = 0$ on S. Since the relation for E_Y in (3) is derived as if σ were present but not **I**, and since σ and σ_1 satisfy B and B_1 respectively, we can apply Theorem 4 of (**14**), and thus $\sigma_1 \equiv 0$. Moreover it is clear that Theorems 2*a*, 4*a*, and 4*b* also apply. Thus we have proved that if **I** and σ satisfy B' and B then there is no line distribution of charge on C and no isolated charge at a vertex of C. This may be restated thus: if **I** and σ satisfy B' and B respectively, if S has a continuously turning normal, and if the component of **I** normal to C, where C has a continuously turning tangent, is finite, then that component is zero. Also, if C is the continuously turning edge on a surface where the direction of the normal changes discontinuously, then the component of **I** normal to C is continuous across C.

It also follows from Theorems 2 and 3 that at Q, where Q is not necessarily on S,

$$|\mathbf{E}_1| = o(\epsilon^{-1}), \qquad |\mathbf{B}_1| = o(\epsilon^{-1}) \tag{4}$$

as $\epsilon \to 0$ where ϵ is the shortest distance of Q from C.

5. The behaviour of E_{ξ_1} and B_{ξ_1}

Copson (**15**) has conjectured that the component of $\mathbf{E}_1$ along C vanishes at C. This is known to be true in a number of solutions, but solutions are also known in which it is not true, e.g. Wiegrefe (**2**). We shall give sufficient conditions for the component to be finite and also give a counter-example to show that the conditions already employed are insufficient. It will be assumed that the results of the preceding section hold.

It is convenient to make a change in notation so that the (x, y, z) coordinates become (x_1, x_2, x_3) and the point of observation Q becomes (X_1, X_2, X_3). The equation of S is $x_3 = f(x_1, x_2)$ and the parametric form of C is $x_1 = G_1(t)$, $x_2 = G_2(t)$, $x_3 = f(x_1, x_2)$ where $G_1(0) = 0$, $G_2(0) = 0$, $G_1'(0) = 1$, $G_2'(0) = 0$, the primes indicating derivatives with respect to the argument. Let ξ_1, ξ_2, ξ_3 be curvilinear coordinates so that S is $\xi_3 = 0$, $\xi_2 > 0$, C is $\xi_3 = 0$, $\xi_2 = 0$, and O is $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$. In particular we may take $x_1 = G_1(\xi_1)$, $x_2 = G_2(\xi_1) + \xi_2$, $x_3 = \xi_3 + f(x_1, x_2)$. Let h_{ij} be defined by

$$h_{ij} = h_{ji} = \sum_k \frac{\partial x_k}{\partial \xi_i} \frac{\partial x_k}{\partial \xi_j}.$$

Let Q be (Ξ_1, Ξ_2, Ξ_3) in the curvilinear coordinates and H_{ij} be the value of h_{ij} at Q.

Suppose that $\sigma(\xi_1, \xi_2)$ satisfies conditions B_2 of (**14**), i.e. in $|\xi_1| \leqslant \xi_1'$, $0 \leqslant \xi_2 \leqslant \xi_2'$, then the following expansion holds:

$$\sigma(\xi_1, \xi_2) = \sigma(\xi_2) + \xi_1 \sigma'(\xi_2) + \xi_1^2 \sigma''(\xi_1, \xi_2)$$

where σ'' is summable on S and $\sigma(\xi_2)$, $\sigma'(\xi_2)$ are both $O[\xi_2^{-1}\{\log(1/\xi_2)\}^{-2-\beta}]$ where $\beta > 0$. (It may be noted that we can replace ξ_2 by ϵ.) Then we may apply Theorem 5 of (**14**) to that part of E_{ξ_1} which is due to σ and hence that part of E_{ξ_1} is finite. There remains the part due to $\mathbf{I}$ to be considered. Apart from multiplicative constants

$$E_{\xi_1} = \int\limits_{-\xi_1'}^{\xi_1'} \int\limits_{0}^{\xi_2'} \left(\sum_k I_{xk} \frac{\partial X_k}{\partial \Xi_1}\right) r^{-1} \sqrt{(h_{11} h_{22} - h_{12}^2)}\, d\xi_1\, d\xi_2 + O(1), \tag{5}$$

where $r^2 = \sum\limits_k (X_k - x_k)^2$.

Let I_{ξ_1}, I_{ξ_2} be the components of the current density along the curves $\xi_2 =$ constant, $\xi_1 =$ constant respectively. Then I'_{ξ_1}, I'_{ξ_2}, the components along and normal to $\xi_2 =$ constant, are given by

$$I'_{\xi_1} = (1/\sqrt{h_{11}}) \sum_k I_{xk} \frac{\partial x_k}{\partial \xi_1} = I_{\xi_1} + \{h_{12}/\sqrt{(h_{11} h_{22})}\} I_{\xi_2},$$
$$I'_{\xi_2} = \sqrt{(1 - h_{12}^2/h_{11} h_{22})}\, I_{\xi_2}.$$

Since we have proved in section 4 that $I'_{\xi_2} \to 0$ as $\xi_2 \to 0$ and since

$$\sqrt{(1 - h_{12}^2/h_{11} h_{22})} \neq 0$$

it follows that $I_{\xi_2} \to 0$ as $\xi_2 \to 0$. Now

$$\left| I'_{\xi_1} \sqrt{h_{11}} - \sum_k I_{xk} \frac{\partial X_k}{\partial \Xi_1} \right| = \left| \sum_k I_{xk} \left(\frac{\partial x_k}{\partial \xi_1} - \frac{\partial X_k}{\partial \Xi_1} \right) \right| \leqslant MR|\mathbf{I}|$$

where $$R^2 = \sum_k (\Xi_k - \xi_k)^2.$$

Since $r^2 \geqslant K_1 R^2$ (from equation (21) of (**14**)) and $\mathbf{I}$ is summable on S we may replace $\sum\limits_k I_{xk}\, \partial X_k/\partial \Xi_1$ in (5) by $I'_{\xi_1} \sqrt{h_{11}}$. Thus the only part of E_{ξ_1} which may be non-finite is the potential of a charge distribution I'_{ξ_1}. Hence if I'_{ξ_1} satisfies B_2 we may apply Theorem 5 of (**14**) and hence this part is $O(1)$. Thus we have proved that if σ satisfies B and B_2, and if $\mathbf{I}$ satisfies B' and I'_{ξ_1} satisfies B_2 then $E_{\xi_1} = O(1)$ in the neighbourhood of C.

Let us now consider B_{ξ_1} which is given by

$$B_{\xi_1} = \frac{1}{4\pi\sqrt{H_{11}}} \int\limits_{-\xi_1'}^{\xi_1'} \int\limits_{0}^{\xi_2'} \left\{ \sum_{i,j,k} A_{ijk} \frac{\partial X_i}{\partial \Xi_1} I_{xj}(X_k - x_k) \right\} \times$$
$$\times r^{-3} \sqrt{(h_{11} h_{22} - h_{12}^2)}\, d\xi_1\, d\xi_2 + O(1), \tag{6}$$

where $A_{ijk} = 0$ if i, j, k are not all different and otherwise $A_{ijk} = \pm 1$ according as i, j, k are in cyclic or anti-cyclic order. Now

$$\left|\frac{\partial X_i}{\partial \Xi_1} I_{x_j} - \frac{\partial X_j}{\partial \Xi_1} I_{x_i} - \frac{1}{\surd h_{22}} I_{\xi_2}\left(\frac{\partial X_i}{\partial \Xi_1}\frac{\partial X_j}{\partial \Xi_2} - \frac{\partial X_j}{\partial \Xi_1}\frac{\partial X_i}{\partial \Xi_2}\right)\right| \leqslant MR\{|I'_{\xi_1}|+|I'_{\xi_2}|\}.$$

Therefore, since I'_{ξ_1} satisfies B_2 and I'_{ξ_2} is a continuous bounded function, we may replace I_{x_j} in the integrand of (6) by $\dfrac{1}{\surd h_{22}} I_{\xi_2} \dfrac{\partial X_j}{\partial \Xi_2}$ and the equation still holds. Further we may replace $X_k - x_k$ by

$$\sum_l \frac{\partial X_k}{\partial \Xi_l}(\xi_l - \Xi_l)$$

since

$$\left|X_k - x_k - \sum_l \frac{\partial X_k}{\partial \Xi_l}(\xi_l - \Xi_l)\right| \leqslant MR^2.$$

Then, since $r^2 \geqslant K_1 R^2$ and $\xi_3 = 0$, $\Xi_1 = 0$

$$|B_{\xi_1}| \leqslant \frac{|J||\Xi_3|}{4\pi\surd H_{11}} \int_{-\xi'_1}^{\xi'_1} \int_0^{\xi'_2} (|I_{\xi_2}|/\surd h_{22}) R^{-3} \surd(h_{11}h_{22} - h_{12}^2)\, d\xi_1\, d\xi_2 + O(1),$$

where $J = \partial(X_1, X_2, X_3)/\partial(\Xi_1, \Xi_2, \Xi_3)$ and $R^2 = \xi_1^2 + (\Xi_2 - \xi_2)^2 + \Xi_3^2$. Now I_{ξ_1} is bounded and zero at $\xi_2 = 0$. Therefore by defining I_{ξ_2} as zero off S we see that the expression on the right is, apart from bounded multiplicative factors, the normal derivative of the potential of a continuous charge distribution which is known to have a finite limit when $\Xi_3 \to 0$. Hence $B_{\xi_1} = O(1)$ in the neighbourhood of C.

If C instead of being the boundary of S is the continuously turning edge where the direction of the normal to S changes discontinuously, we may apply the above analysis in turn to the portions separated by C. The only difference is that I_{ξ_2} is non-zero but continuous across C. Hence in this case $E_{\xi_1} = O(1)$, $B_{\xi_1} = O(1)$. Clearly the same is true of the appropriate components of $\mathbf{E}_1$ and $\mathbf{B}_1$, since $\sigma_1 \equiv 0$.

As a counter-example we may consider the case when S is plane and C a straight line so that ξ_1, ξ_2, ξ_3 are Cartesian coordinates. Take

$$I_{\xi_1} = 0, \qquad I_{\xi_2} = -i\omega\xi_1/\{\log(1/\xi_2)\}, \qquad \sigma = \xi_1/[\xi_2\{\log(1/\xi_2)\}^2].$$

Then the only non-finite contribution to E_{ξ_1} is from σ and it follows from the counter-example in section 8 of (**14**) that this becomes infinite like $\log\log(1/\Xi_2)$. Thus conditions B and B' are insufficient to ensure the finiteness of E_{ξ_1}.

6. Uniqueness for the edge with continuously turning tangent

We have now obtained sufficient conditions for the solution (1) to be unique. Let S_1 be a small cylinder of radius δ surrounding the edge. On S_1, $\mathbf{E}_1 \wedge \mathbf{B}_1^* . \mathbf{n} = E_{1\xi_1} B_{11}^* - B_{1\xi_1}^* E_{11}$, where E_{11} and B_{11}^* are linear functions with bounded coefficients of the components of $\mathbf{E}_1$ and $\mathbf{B}_1^*$ respectively. From sections 4 and 5 it follows that, if σ satisfies B and B_2 and if $\mathbf{I}$ satisfies B' and I'_{ξ_1} satisfies B_2, then $E_{1\xi_1} B_{11}^* - B_{1\xi_1}^* E_{11} = o(\delta^{-1})$. Since the radius of S_1 is δ we have $\int_{S_1} \mathbf{E}_1 \wedge \mathbf{B}_1^* . d\mathbf{S} = o(1)$ as $\delta \to 0$ and hence the field is determined uniquely by (1), assuming that the appropriate radiation conditions at infinity are satisfied. We are referring here to the case when S and C are finite; when they are infinite restrictions on the behaviour of the charge and current densities at infinity have to be introduced to ensure a unique solution. In view of the last paragraph but one of the preceding section it is immaterial whether C is an edge or the boundary of S.

One consequence of this result is that there is only one solution differentiable on S such that $\sigma = O(\epsilon^{\alpha_1})$, $I'_{\xi_1} = O(\epsilon^{\alpha'_1})$, $I'_{\xi_2} = O(1)$ where $\alpha_1 > -1$, $\alpha'_1 > -1$, since such a solution clearly satisfies the conditions imposed. This is the type of solution in which one would probably be interested when solving a practical problem.

7. Two-dimensional fields

The behaviour of two-dimensional fields is similar to that in three dimensions, but since the conditions are simpler it seems worth while to state them separately. Let C_1 be a regular curve with end-points F_1 and F_2, and consider the behaviour near F_1. Suppose that the fields are independent of the x-coordinate when the notation of sections 1–4 is used. Then

$$\left.\begin{aligned} \mathbf{E}_1 &= \mathbf{E}_0 + \frac{i\omega}{4\pi k^2} \int_{C_1} (-k^2\psi\mathbf{I} - i\omega\sigma \operatorname{grad}\psi)\, ds - \frac{i\omega}{4\pi k^2}\sigma_1 \operatorname{grad}\psi \\ \mathbf{B}_1 &= \mathbf{B}_0 + \frac{1}{4\pi} \int_{C_1} (\mathbf{I} \wedge \operatorname{grad}\psi)\, ds \end{aligned}\right\} \tag{7}$$

where $\psi = -\pi i H_0^{(2)}(kr)$, $r^2 = (y_1-y)^2 + (z_1-z)^2$, $H_0^{(2)}$ is the Hankel function of the second kind and in the σ_1 term r is measured from F_1. Formulae (7) satisfy Maxwell's equations if (i) it is permissible to take derivatives under the integral sign, (ii) it is permissible to integrate $\int_{C_1} \sigma\psi\, ds$, $\int_{C_1} \sigma \operatorname{grad}\psi\, ds$ by parts, σ being the quantity integrated, and (iii) $\sigma_1 = I_y$ at F. These conditions can obviously be satisfied. We can replace ψ in (7) by $-2\log r$ without making the expressions less finite. Then $\mathbf{E}_1$ and $\mathbf{B}_1$ are obtained from the potentials and intensities of certain charge distributions.

Let conditions B (i)′, (ii)′, B' (i)′, (ii)′ be the same as B (i), (ii), B' (i), (ii) except that S is replaced by C_1 and C by F_1. Then, if σ and the components of $\mathbf{I}$ satisfy B (i)′, (ii)′, (iii) and B' (i)′, (ii)′, (iii) respectively, Theorem 6 of (**14**) shows that

$$\mathbf{E}_1 = o(\epsilon^{-1}), \qquad \mathbf{B}_1 = o(\epsilon^{-1}). \tag{8}$$

Then Theorem 7 of (**14**) shows that $\sigma_1 = 0$ at F_1. Similar results hold at F_2 and at a point where the direction of C_1 changes discontinuously. Hence we have shown that if σ and $\mathbf{I}$ satisfy B (i)′, (ii)′, (iii) and B' (i)′, (ii)′, (iii), and if the current density perpendicular to the x-axis is finite it is zero at an end-point of C_1 and continuous across a point where the direction of C_1 changes discontinuously.

The discussion corresponding to section 5 is considerably simpler. The possible non-finite part of E_x is the potential of the distribution I_x. If $I_x = O[\epsilon^{-1}\{\log(1/\epsilon)\}^{-2-\beta}]$ this part is finite and $E_x = O(1)$. It is easy to show that B_x is mainly the normal derivative of the potential of the current density component perpendicular to the x-axis and hence $B_x = O(1)$. It now follows by a use of (8) that, with suitable conditions at infinity, there is only one solution of the type (7) if σ satisfies B (i)′, (ii)′, (iii), if $\mathbf{I}$ satisfies B' (i)′, (ii)′, (iii), and if, in addition, I_x satisfies the condition just given above.

8. Some known solutions

In this section we briefly indicate how various known solutions agree with our conditions. Before doing so it is convenient to remark that B (ii) is satisfied with $\alpha = 1$ if $\max(|\partial\sigma/\partial x|, |\partial\sigma/\partial y|) = o(\epsilon^{-2})$ in $r_1 < a\epsilon$.

In the Sommerfeld solution for the two-dimensional diffraction by a semi-infinite plane we have $\sigma = O(y^{-\frac{1}{2}})$ taking the plane as $z = 0$, $y \geqslant 0$. Since

$$\partial\sigma/\partial y = O(y^{-\frac{3}{2}}) = o(y^{-2}),$$

B (ii)′ is satisfied. The other conditions are clearly satisfied. Similarly for the current densities.

The Wiegrefe (**2**) solution for the three-dimensional diffraction by a semi-infinite plane does not satisfy B' (iv) or B_2 since the current density parallel to the edge is $O(y^{-\frac{3}{2}})$. On the other hand, the solution given by the author (**3**) does, since the current density parallel to the edge is $y^{-\frac{1}{2}}e^{-ikx\cos\theta}+O(y^{\frac{1}{2}})$. The formulae given by the author agree with those obtained for a special case by Copson (**16**).

In the solution of the diffraction by a circular disk given by Meixner and Andrejewski (**5**) we have, taking polar coordinates ρ', ϕ' on the disk, $I_{\rho'} = (1-\rho'^2)^{\frac{1}{2}}\cos\phi'$, $I_{\phi'} \approx (1-\rho'^2)^{-\frac{1}{2}}\sin\phi'$, $\sigma \approx \rho'(1-\rho'^2)^{-\frac{1}{2}}\cos\phi'$. Since the derivative parallel to the edge is $\partial/\partial\phi'$ there is no difficulty in seeing that the conditions are satisfied. Möglich's (**4**) solution fails to satisfy B (iv) or B_2 since the charge density is $O\{(1-\rho'^2)^{-\frac{3}{2}}\}$ near the edge $\rho' = 1$.

9. Behaviour near a corner

By a corner we mean the neighbourhood of a vertex O where the direction of C changes discontinuously but where S has a continuously turning normal. This definition excludes the type of corner that occurs at the end of a rectangular wave-guide radiating into space, but we shall extend our results to cover this case.

We have already shown in section 4 that, when C has a vertex at O, the field is $o(\epsilon^{-1})$ and that there is neither a line distribution on C nor an isolated charge at O. However, the uniqueness theorem of section 6 cannot be deduced at once since the analysis of section 5 breaks down in the neighbourhood of a vertex. Here we prove uniqueness in the simple case when the neighbourhood of O is plane and bounded by two straight lines and when the current and charge densities have certain simple forms.

We suppose that Q is near one of the straight lines and that, with this line as polar axis, the spherical polar coordinates of Q are (ρ, θ, ϕ), S being $\phi = 0$. Let the angle between the straight lines which contains S be θ' which, for the moment, will be taken to be less than π. The fields from parts of S which are a distance from $\theta = 0$ greater than d, where d is fixed but small, will be finite. Consider now the potential of a charge density $\rho_1^{\lambda}\Theta$ in $0 \leqslant \rho_1 \leqslant d$, $0 \leqslant \theta_1 \leqslant \theta'$, where Θ is a function of θ_1 only and Θ behaves like θ_1^{ν} as $\theta_1 \to 0$. A similar behaviour is assumed near $\theta_1 = \theta'$. In order to comply with B (iv), i.e. that the charge on a line (not $\theta_1 = 0$ or $\theta_1 = \theta'$) shall be finite, we require that $\lambda > -1$ and $\nu > -1$. The potential is, apart from multiplicative constants,

$$\rho^{\lambda+1} \int_0^{d/\rho} \int_0^{\theta'} \rho_1^{\lambda+1} \Theta r^{-1}\, d\rho_1\, d\theta_1$$

where $r^2 = \rho_1^2 - 2\rho_1(\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos\phi) + 1$. Let T be a large number but such that $T < d/\rho$. For $T \leqslant \rho_1 \leqslant d/\rho$, r^{-1} may be expanded in inverse powers of ρ_1 to give, when λ is not an integer, $U + V\rho^{\lambda+1}$ where U is a series of non-negative powers of ρ, $\cos\theta$, and $\sin\theta\cos\phi$ and V is a series of non-negative powers of $\cos\theta$ and $\sin\theta\cos\phi$. For $0 \leqslant \rho_1 \leqslant T$ when ρ_1 is not near unity and θ_1 not near θ the integrand is finite and the contribution of the integral is U_1 (say). There remains the range $1 - \eta_1 \leqslant \rho_1 \leqslant 1 + \eta_1$ $(\eta_1 > 0)$, $0 \leqslant \theta_1 \leqslant \theta''$ where η_1 and θ'' are small. Expand $\rho_1^{\lambda+1}$ in a power series about $\rho_1 = 1$; the first term is

$$\rho^{\lambda+1} \int_{1-\eta_1}^{1+\eta_1} \int_0^{\theta''} \Theta r^{-1}\, d\rho_1\, d\theta_1 = \rho^{\lambda+1} \int_0^{\theta''} \Theta \log(R_2/R_3)\, d\theta,$$

where R_2, R_3 are the values of $\rho_1 - \cos\theta\cos\theta_1 - \sin\theta\sin\theta_1\cos\phi + r$ when

$\rho_1 = 1+\eta_1$, $1-\eta_1$ respectively. Since η_1 is fixed and θ, θ'' small the only part of the integrand which is singular is that involving R_3. From that part, putting $\Theta = \theta_1^{\nu}$, we have, as in reference (**3**),

$$\rho^{\lambda+1}\int\limits_0^{\theta''} \theta_1^{\nu}\log(\theta^2+\theta_1^2-2\theta\theta_1\cos\phi)\,d\theta_1+W\rho^{\lambda+1}$$
$$= \frac{2\pi\rho^{\lambda+1}\theta^{\nu+1}\cos\{(\nu+1)(\phi-\pi)\}}{\sin(\nu+1)\pi}+W\rho^{\lambda+1}$$

when ν is not an integer. Here W is a series of non-negative powers of θ and $\cos\phi$. The remaining terms in the expansion of $\rho_1^{\lambda+1}$ give terms which can be included in W and terms involving $\theta^{\nu+3}$,.... Hence the potential of the charge distribution is

$$U+(U_1+V+W)\rho^{\lambda+1}+\rho^{\lambda+1}W\theta^{\nu+3}+\frac{2\pi\rho^{\lambda+1}\theta^{\nu+1}\cos\{(\nu+1)(\phi-\pi)\}}{\sin(\nu+1)\pi}. \quad (9)$$

Using this expression we now deduce the field due to the charge density $\rho_1^{\lambda}\Theta$ and the current density whose radial and transverse components are (const)$\rho_1^{\lambda_1}\Theta_1$, (const)$\rho_1^{\lambda_2}\Theta_2$ where $\Theta_1 = \theta_1^{\nu_1}$, $\Theta_2 = \theta_1^{\nu_2}$ as $\theta_1 \to 0$. In order that the current density normal to $\theta_1 = 0$ should vanish we require $\nu_2 > 0$. In order that the total current on a line should be finite we require $\lambda_1 > -1$, $\lambda_2 > -1$, $\nu_1 > -1$. Since the charge density can be derived from the current density by the equation of continuity, λ_1 and λ_2 can be less than zero only if $\lambda_1 = \lambda_2$, $\nu_1 = \nu_2-1$ and a certain relation holds between the multiplying constants. We consider the case when λ, λ_1, λ_2, ν, ν_1 all lie in $(-1, 0)$ so that the fields may not be finite. It is easy to verify that the use of Stokes's theorem required in section 1 is permissible for these charge and current densities.

To obtain the field derivatives of the expression (9) are taken and here we note that

$$\frac{\partial U}{\partial\rho},\ \frac{1}{\rho\theta}\frac{\partial U}{\partial\phi},\ \frac{1}{\rho}\frac{\partial U}{\partial\theta},\ \frac{1}{\theta}\frac{\partial V}{\partial\phi},\ \frac{1}{\theta}\frac{\partial W}{\partial\phi},\ \frac{1}{\theta}\frac{\partial U_1}{\partial\phi}$$

are finite for all ρ, θ near the origin. After that remark a tedious, but straightforward, calculation shows that

$$\left.\begin{aligned} E_{1\rho} &= O(\rho^{\lambda}), & E_{1\theta} &= O(\rho^{\lambda}\theta^{\nu}), & E_{1\phi} &= O(\rho^{\lambda}\theta^{\nu}) \\ B_{1\rho} &= O(\rho^{\lambda_1}+\rho^{\lambda_2}), & B_{1\theta} &= O(\rho^{\lambda_1}\theta^{\nu_1}+\rho^{\lambda_2}), & B_{1\phi} &= O(\rho^{\lambda_1}\theta^{\nu_1}+\rho^{\lambda_2}) \end{aligned}\right\}. \quad (10)$$

Consider the integral of the Poynting vector over a sphere of radius ρ_0 with centre O. Let θ_0 (> 0) be small and fixed. Since $\mathbf{E}_1 = o(\epsilon^{-1})$, $\mathbf{B}_1 = o(\epsilon^{-1})$ from (4) we have for $\theta'-\theta_0 \geqslant \theta \geqslant \theta_0$, $\theta \geqslant \theta'+\theta_0$ that $\mathbf{E}_1 = o(\rho_0^{-1})$, $\mathbf{B}_1 = o(\rho_0^{-1})$ as $\rho_0 \to 0$, so that the integral of the Poynting vector over this

portion is $o(1)$ as $\rho_0 \to 0$. For $\theta_0 \geqslant \theta$ we see by using (10) that this portion of the sphere contributes $O\{\rho_0^{\lambda+2}\theta_0^{\nu+2}(\rho_0^{\lambda_1}\theta_0^{\nu_1}+\rho_0^{\lambda_2})\} = o(1)$ as $\rho_0 \to 0$ for fixed θ_0. Similarly for the neighbourhood of the other edge.

Now take the Poynting vector over the surface of the cone $\theta = \delta/d'$ $(< \theta_0)$, $\rho_0 \leqslant \rho \leqslant d'$ where d' $(\ll d)$ is fixed. It is, using (10),

$$O\{\rho_0^{\lambda+\lambda_1+2}(\delta^{\nu+1}+\delta^{\nu_1+1})+\rho_0^{\lambda+\lambda_2+2}(\delta^{\nu+1}+\delta)+\delta(\delta^{\nu}+\delta^{\nu_1}+1)\} = o(1) \quad (11)$$

as ρ_0 and δ tend to zero. Similarly for the other edge. The edge and corner may now be surrounded by the sphere $\rho = \rho_0$, the cone $\theta = \delta/d'$, $\rho_0 \leqslant \rho \leqslant d'$, a similar cone round the other edge, and the cylinder of radius δ and centre the edge for $\rho \geqslant d'$. From the above it follows that as $\rho_0 \to 0$ and $\delta \to 0$ the integral of the Poynting vector over this surface tends to zero; hence if appropriate conditions are satisfied at infinity the solution is unique.

There is clearly no difficulty in obtaining the same result when the angle of the corner is greater than π but less than 2π. The result also extends to current and charge densities which are the sums of products of the type $\rho^{\lambda}\Theta$. We may also remove the restriction that λ and ν are not integers. For if ν is an integer s $(\geqslant 0)$ the term involving $\theta^{\nu+1}$ in (9) is replaced by

$$\theta^{s+1}\log\theta = o(\theta^{\nu+1})$$

for $\nu < 0$ and hence this is included in the above. If λ is an integer L $(\geqslant 0)$ the only difference is that U contains a term $\rho^{L+1}\log\rho = o(\rho^{\lambda+1})$ $(\lambda < 0)$ so that this, too, is included.

Also we note that the form of the coefficient of $\theta^{\nu+1}$ in (9) enables us to show that ν, ν_1, and ν_2 are odd half-integers as in reference (**3**).

Finally we extend the results to the corner that occurs at the end of a rectangular wave-guide by treating each of the flat portions separately as above. There is no alteration for the free edges. For the common edge the only difference is that ν_2 may be zero; when ν_2 is not zero there is no difference. When ν_2 is zero th alteration in (10) is that $B_{1\rho} = O(\rho^{\lambda_1}+\rho^{\lambda_2}\log\theta)$ and this changes the terms $\rho_0^{\lambda+\lambda_2+2}\delta^{\nu+1}$, $\delta^{\nu+1}$ in (11) into $\rho_0^{\lambda+\lambda_2+2}\delta^{\nu+1}\log\delta$, $\delta^{\nu+1}\log\delta$ respectively. Since the new terms are $o(1)$ our conclusions remain valid.

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THE INDUCTION OF ELECTRIC CURRENTS IN A UNIFORMLY CONDUCTING CIRCULAR DISK BY THE SUDDEN CREATION OF MAGNETIC POLES

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SUMMARY

Using toroidal coordinates, a double space of two regions is constructed in the manner of Sommerfeld's theory. The Riemann potential for a single pole is obtained and Jeans's treatment of uniform finite plane current sheets is applied to the case of the circular disk.

1. Introduction

THE induction of electric currents in a uniformly conducting infinite plane sheet was discussed by Maxwell (**1**). Jeans (**2**) extended the theory to uniform finite plane sheets, using the method of multiform potentials due to Sommerfeld (**3**). He discussed two special cases, namely the semi-infinite plane and the infinite strip of uniform width.

In the present paper Jeans's theory is applied to the case of a uniformly conducting circular disk. The theory is briefly described in section 2. In section 3 the Riemann potential for a single pole is obtained and the solution for the disk, together with a numerical illustration, is given in section 4.

2. Jeans's theory

Following Maxwell, Jeans's method of solution consisted in first determining the currents induced by a sudden change in the external field and then dealt with the decay of these currents under the action of resistance and self-induction.

In applying Sommerfeld's method of multiform potentials, a Riemann space of two regions is first constructed. The first region is to be identical with the space containing the magnetic system and the second will contain certain images of this system. The field of these images will represent the induced field. The current sheet itself must be the branch membrane of the space and its boundary the branch line. For every point (x, y, z) in the first region there will be a corresponding point (x', y', z') in the second, where $x = x'$, $y = y'$, $z = z'$, if the sign $=$ is taken only to mean the algebraic equality of the magnitude of the quantities which it separates. The plane of the sheet will always be taken as the z-plane.

The required Riemann potential of a single magnetic pole situated at

(x, y, z) must satisfy the following conditions: (i) it must satisfy Laplace's equation; (ii) it must be *single-valued*, continuous and finite in the *double* space, except in the neighbourhood of (x, y, z) in the first space, where it becomes infinite like R^{-1}, R being the distance between the pole and the field point; it does not become infinite at the congruent point (x', y', z') in the second space; (iii) it must vanish at infinity. Unlike the Newtonian potential, the Riemann potential does not have the same values at corresponding points in the two regions of the double space.

Sommerfeld showed that the Newtonian potential due to a single pole at (x, y, z) is equal to the sum of the Riemann potentials of two similar poles, one at (x, y, z) and the other at (x', y', z').

In electrostatic and hydrodynamic problems, natural boundaries in the form of finite plane sheets can be reduced to infinite plane boundaries, provided the Riemann potential is used. It is possible, then, to apply the method of images. The images, however, should be in the second region of the double space.

Let $\Omega^{(e)}$ be the magnetic potential of a simple unit pole at (x_0, y_0, z_0) and $P(x_0, y_0, z_0)$ its Riemann potential, and let $\Omega^{(i)}$ be the magnetic potential of the currents induced in the sheet by the sudden introduction of the pole. Maxwell pointed out that the determination of $\Omega^{(i)}$ just after the sudden change in the external field is a hydrodynamic problem. Jeans used this fact to prove that just after the introduction of the pole we must have:

$$\Omega^{(i)} = P(x_0', y_0', -z_0') - P(x_0', y_0', z_0'), \tag{1}$$

$$\frac{\partial \Omega^{(i)}}{\partial z} = -P_1(x_0', y_0', -z_0') + P_1(x_0', y_0', z_0'), \tag{2}$$

where $P_1(x_0, y_0, z_0)$ is the Riemann potential due to a unit doublet at (x_0, y_0, z_0) with axis along the z-axis.

At any subsequent time t, while the currents are decaying, Jeans found that

$$\frac{\partial \Omega^{(i)}}{\partial z} = P_1(x_0', y_0', z_0' + vt) - P_1(x_0', y_0', -z_0' - vt), \tag{3}$$

where $v = \sigma/2\pi$ and σ is the surface resistance of the sheet. This can be expressed by saying that the image system representing the induced field moves away from the sheet with velocity v.

3. The Riemann space for the disk

In forming the Riemann space for the circular disk, toroidal† coordinates (ψ, σ, ϕ) will be used. The surfaces ψ = constant are the spherical caps having the circle $z = 0$, $\rho = a$ (in cylindrical coordinates) in common. The upper and lower surfaces of the disk whose boundary is this circle are given

† Bateman (**4**) gives the geometrical interpretations of these coordinates.

by $\psi = \pi$, $-\pi$ respectively, while the z-axis is given by $\sigma = 0$. It will be noticed that the value of ψ suffers a discontinuous change of 2π in crossing the 'principal disk'.

Consider now a double space of three dimensions with two superposed regions as explained above. The passage from one region to the other is made when a point passes through the principal disk. The two regions may be distinguished from each other by assigning different ranges of values to ψ, namely $-\pi < \psi < \pi$ in the first region, and $\pi < \psi < 3\pi$ in the second. Corresponding to a point (ψ, σ, ϕ) in the first region, there is an associated point $(\psi+2\pi, \sigma, \phi)$ in the second.

We shall regard our conducting disk as the principal disk of the system and it will be assumed that it is situated in the first region of the double space.

The ordinary Newtonian potential at (ψ, σ, ϕ) due to a simple magnetic pole of unit strength situated at $(\psi_0, \sigma_0, \phi_0)$ is given by R^{-1}. This potential can be expressed (Bateman (4)) in terms of toroidal coordinates in the form

$$\sqrt{2}\,\pi a R^{-1} = (\cosh\sigma - \cos\psi)^{\frac{1}{2}}(\cosh\sigma_0 - \cos\psi_0)^{\frac{1}{2}} \int\limits_{\alpha}^{\infty} \frac{\sinh u(\cosh u - \cosh\alpha)^{-\frac{1}{2}}}{\cosh u - \cos(\psi-\psi_0)}\, du, \tag{4}$$

where $$\cosh\alpha = \cosh\sigma\cosh\sigma_0 - \sinh\sigma\sinh\sigma_0\cos(\phi-\phi_0). \tag{5}$$

Using the identity

$$2\sinh u\{\cosh u - \cos(\psi-\psi_0)\}^{-1}$$
$$= \sinh\tfrac{1}{2}u[\{\cosh\tfrac{1}{2}u - \cos\tfrac{1}{2}(\psi-\psi_0)\}^{-1} + \{\cosh\tfrac{1}{2}u + \cos\tfrac{1}{2}(\psi-\psi_0)\}^{-1}], \tag{6}$$

it is found that (Bateman (4))

$$R^{-1} = P(\psi_0, \sigma_0, \phi_0) + P(\psi_0+2\pi, \sigma_0, \phi_0), \tag{7}$$

where $$P(\psi_0, \sigma_0, \phi_0) = R^{-1}[\tfrac{1}{2} + \pi^{-1}\sin^{-1}\{\cos\tfrac{1}{2}(\psi-\psi_0)\operatorname{sech}(\tfrac{1}{2}\alpha)\}]. \tag{8}$$

It can be shown that $P(\psi_0, \sigma_0, \phi_0)$ is a solution of Laplace's equation when considered as a function of either $(\psi_0, \sigma_0, \phi_0)$ or (ψ, σ, ϕ). In fact it is symmetrical in the two sets of coordinates. Moreover, it is a uniform function of (ψ, σ, ϕ) in the *double space*, since its value is not altered when ψ is increased by 4π. It is also continuous in the double space except at the point $(\psi_0, \sigma_0, \phi_0)$, where it has an infinity of order R^{-1} (it is finite at $(\psi_0+2\pi, \sigma_0, \phi_0)$, the corresponding point in the second region of the double space at which R^{-1} is infinite). It can thus be seen that $P(\psi_0, \sigma_0, \phi_0)$ has all the properties of the Riemann potential of a single pole at $(\psi_0, \sigma_0, \phi_0)$. It will also be seen from (7) that the Newtonian potential for a pole at a point in the first region is equal to the Riemann potential of two similar poles, the first at the same point and the second at the corresponding point in the second region, which is true in general for the Riemann potential as stated above.

4. Solution for the disk

Jeans's method can now be applied without change in the symbolical notation, except that functions of (x, y, z) will be functions of (ψ, σ, ϕ). If at time $t = 0$ a single pole is created at the point $(\psi_0, \sigma_0, \phi_0) \equiv (z_0, \rho_0, \phi_0)$, then just after the creation of the pole

$$\Omega^{(i)} = P(-\psi_0+2\pi, \sigma_0, \phi_0) - P(\psi_0+2\pi, \sigma_0, \phi_0). \tag{9}$$

It will be seen from (8), (9) that the current function, which is given by $\Omega^{(i)}/2\pi$ at the surface of the disk, vanishes at its boundary $\sigma = \infty$, as was to be expected. Similarly,

$$\frac{\partial\Omega^{(i)}}{\partial z} = -P_1(-\psi_0+2\pi, \sigma_0, \phi_0) + P_1(\psi_0+2\pi, \sigma_0, \phi_0) \quad \text{at } t = 0. \tag{10}$$

These two images will represent $\partial\Omega^{(i)}/\partial z$ at any following time provided they move with velocity v in the direction of the normals away from the disk. This means that the expression on the right-hand side of (10) will represent $\partial\Omega^{(i)}/\partial z$ at time t provided the point $(\psi_0, \sigma_0, \phi_0)$ is identified with the point $(z_0+vt, \rho_0, \phi_0) \equiv (\bar{z}_0, \rho_0, \phi_0)$. Now

$$\left.\begin{aligned} P_1(\psi_0, \sigma_0, \phi_0) &= \frac{\partial}{\partial z_0} P(\psi_0, \sigma_0, \phi_0) = \frac{\partial}{\partial z_0}\{R^{-1}F(\psi_0, \sigma_0, \phi_0)\} \\ \text{where} \quad F(\psi_0, \sigma_0, \phi_0) &= \tfrac{1}{2}+\pi^{-1}\sin^{-1}\{\cos\tfrac{1}{2}(\psi-\psi_0)\operatorname{sech}\tfrac{1}{2}\alpha\} \end{aligned}\right\}. \tag{11}$$

Hence
$$P_1(\psi_0, \sigma_0, \phi_0) = R^{-1}\left(\frac{\partial F}{\partial \psi_0}\frac{\partial \psi_0}{\partial z_0} + \frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial \sigma_0}\frac{\partial \sigma_0}{\partial z_0}\right) + F\frac{\partial R^{-1}}{\partial z_0}. \tag{12}$$

From the relation between toroidal and cylindrical coordinates,

$$z_0+i\rho_0 = a\cot\tfrac{1}{2}(\psi_0+i\sigma_0), \tag{13}$$

it is found, by differentiation, that

$$\left.\begin{aligned} a\frac{\partial\psi_0}{\partial z_0} &= \cos\psi_0\cosh\sigma_0-1 \\ a\frac{\partial\sigma_0}{\partial z_0} &= -\sin\psi_0\sinh\sigma_0 \end{aligned}\right\}. \tag{14}$$

Hence

$$\begin{aligned} P_1(\psi_0, \sigma_0, \phi_0) = {} & -R^{-2}(z_0-z)P(\psi_0, \sigma_0, \phi_0) + \\ & +(2\pi a R)^{-1}[\{(\cosh\sigma_0\cos\psi_0-1)\operatorname{sech}\tfrac{1}{2}\alpha\sin\tfrac{1}{2}(\psi-\psi_0) + \\ & \quad +\tfrac{1}{2}\sin\psi_0\sinh\sigma_0\operatorname{sech}^3\tfrac{1}{2}\alpha\cos\tfrac{1}{2}(\psi-\psi_0)\times \\ & \quad\quad \times(\sinh\sigma_0\cosh\sigma-\cosh\sigma_0\sinh\sigma\cos(\phi-\phi_0))\}\times \\ & \quad\quad\quad \times\{1-\cos^2\tfrac{1}{2}(\psi-\psi_0)\operatorname{sech}^2\tfrac{1}{2}\alpha\}^{-\frac{1}{2}}]. \end{aligned} \tag{15}$$

It will be seen from the above that the expression for $P_1(\psi_0, \sigma_0, \phi_0)$ is rather complicated. If, however, the pole is created on the z-axis at distance z_0 from the disk, this expression simplifies considerably. In this case

$$\sigma_0 = 0, \qquad \psi_0 = 2\tan^{-1}(a/z_0), \qquad \alpha = \sigma, \tag{16}$$

and

$$P_1(\psi_0, 0, \phi_0) = -R^{-3}(z_0-z)[\tfrac{1}{2}+\pi^{-1}\sin^{-1}\{\cos\tfrac{1}{2}(\psi-\psi_0)\operatorname{sech}\tfrac{1}{2}\sigma\}]- \\ -\frac{a}{\pi R}\sin\tfrac{1}{2}(\psi-\psi_0)(z_0^2+a^2)^{-1}\{\cosh^2\tfrac{1}{2}\sigma-\cos^2\tfrac{1}{2}(\psi-\psi_0)\}^{-\frac{1}{2}}, \quad (17)$$

where R, the distance between the field point (ψ, σ, ϕ) and the point $(\psi_0, \sigma_0, \phi_0)$, is given by

$$R^2 = (z_0-z)^2+\rho^2. \quad (18)$$

If R_1 is the distance between the field point and the point $(-\psi_0, \sigma_0, \phi_0)$, then

$$R_1^2 = (z_0+z)^2+\rho^2. \quad (19)$$

Using (10), it is found that

$$\frac{\partial\Omega^{(i)}}{\partial z} = -[R^{-3}(z_0-z)\{\tfrac{1}{2}-\pi^{-1}\sin^{-1}(\cos\tfrac{1}{2}(\psi-\psi_0)\operatorname{sech}\tfrac{1}{2}\sigma)\}+ \\ +R_1^{-3}(z_0+z)\{\tfrac{1}{2}-\pi^{-1}\sin^{-1}(\cos\tfrac{1}{2}(\psi+\psi_0)\operatorname{sech}\tfrac{1}{2}\sigma)\}+ \\ +(a/\pi)(a^2+z_0^2)^{-1}\{-R^{-1}\sin\tfrac{1}{2}(\psi-\psi_0)(\cosh^2\tfrac{1}{2}\sigma-\cos^2\tfrac{1}{2}(\psi-\psi_0))^{-\frac{1}{2}}+ \\ +R_1^{-1}\sin\tfrac{1}{2}(\psi+\psi_0)(\cosh^2\tfrac{1}{2}\sigma-\cos^2\tfrac{1}{2}(\psi+\psi_0))^{-\frac{1}{2}}\}] \quad (20)$$

at $t = 0$.

The same expression on the right-hand side of (20) will represent $\partial\Omega^{(i)}/\partial z$ at any following time provided R, R_1, and ψ_0 are changed to $\bar{R}$, $\bar{R}_1$, and $\bar{\psi}_0$ respectively, where

$$\left.\begin{aligned} \bar{R}^2 &= (z_0+vt-z)^2+\rho^2 \\ \bar{R}_1^2 &= (z_0+vt+z)^2+\rho^2 \\ \bar{\psi}_0 &= 2\tan^{-1}(a/z_0+vt) \end{aligned}\right\}. \quad (21)$$

The potential of the induced field can thus be evaluated in the form

$$\Omega^{(i)} = -\int_z^\infty \frac{\partial\Omega^{(i)}}{\partial z}\,dz, \quad (22)$$

and the current function as

$$\Phi = \frac{1}{2\pi}\Omega^{(i)}_{(z=0)} = -\frac{1}{2\pi}\int_0^\infty \frac{\partial\Omega^{(i)}}{\partial z}\,dz. \quad (23)$$

If the point at which we are evaluating the field lies also on the z-axis, then $\sigma = 0$ and $\psi = 2\tan^{-1}(a/z)$. In this case using equations (20) and (21) we get

$$-\pi\frac{\partial\Omega^{(i)}}{\partial z} = \tfrac{1}{2}(\psi-\bar{\psi}_0)(\bar{z}_0-z)^{-2}+\tfrac{1}{2}(\psi+\bar{\psi}_0)(\bar{z}_0+z)^{-2}- \\ -a(a^2+\bar{z}_0^2)^{-1}\{(\bar{z}_0-z)^{-1}-(\bar{z}_0+z)^{-1}\}. \quad (24)$$

A special numerical case is now considered. A unit pole is created on the z-axis at distance $2a$ from the origin at time $t = 0$. The normal component of the induced field is calculated at a point on the z-axis at distance a from the

origin using (24). The results are given in the following table. They show that the normal component of the induced field is reduced approximately to $\frac{1}{2}$, $\frac{1}{10}$, and $\frac{1}{60}$ of its initial value after times $2\pi a/\sigma$, $9{\cdot}8\pi a/\sigma$, and $30\pi a/\sigma$ seconds respectively.

TABLE

The normal component of the induced field due to the sudden creation of a unit magnetic pole at $(0, 0, 2a)$ *calculated at* $(0, 0, a)$

$\frac{\sigma}{\pi a}t$	$-\pi\frac{\partial\Omega^{(i)}}{\partial z}$	$\frac{\sigma}{\pi a}t$	$-\pi\frac{\partial\Omega^{(i)}}{\partial z}$	$\frac{\sigma}{\pi a}t$	$-\pi\frac{\partial\Omega^{(i)}}{\partial z}$
0·00	0·3272	5·50	0·0671	12·00	0·0242
0·50	0·2678	6·00	0·0609	14·00	0·0192
1·00	0·2226	6·50	0·0554	16·00	0·0156
1·50	0·1876	7·00	0·0506	20·00	0·0108
2·00	0·1601	7·50	0·0465	24·00	0·0080
2·50	0·1382	8·00	0·0427	30·00	0·0054
3·00	0·1203	8·50	0·0393		
3·50	0·1055	9·00	0·0366		
4·00	0·0934	9·50	0·0338		
4·50	0·0832	10·00	0·0316		
5·00	0·0746	11·00	0·0277		

This problem shows that it is possible to obtain numerical results by the method of this paper. Jeans (**2**) in his discussion of the semi-infinite plane and the infinite strip of uniform width did not give any numerical results.

For any other inducing magnetic system the induced field can be obtained by superposition of the fields of its component poles. Also if we have a *continuously changing* magnetic inducing system, then (following Maxwell (**1**) and Jeans (**2**)) we can calculate the induced field by integration in some cases by letting the change and interval during which it occurs both tend to zero.

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